The Potential Analogue Method of Network Synthesis

By SIDNEY DARLINGTON

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A general method is developed for designing networks with assigned gain or phase characteristics. It is based on the analogy between the gain and phase of linear networks and two-dimensional potential and stream functions, produced by charges corresponding to the network singularities. These analogies exist because the gain and phase functions are the real and imaginary parts of analytic functions of a complex frequency variable. Potential theory is used here to determine charge arrays which correspond to physical network singularities and also yield approximations to assigned potential or stream functions.

1. Introduction

THE problem of network synthesis is the inverse of the much simpler problem of network analysis. If an exponential input voltage, $E \exp (pt)$, is applied to a given network consisting of a finite number of lumped linear elements, we can always calculate the corresponding output voltage, $V \exp (pt)$, in terms of the network constants. Then we define a transmission function $F(p)$ as the logarithm of the ratio V/E. In general $F(p)$ is an analytic function in the complex p-plane. Its value on the real frequency axis, $p = i\omega$, defines the gain and the phase shift of the network.

In the inverse problem we start with an assigned transmission function $F(p)$ and are required to find a network for which $F(p)$ is the transmission function. More frequently we have to design a network with assigned gain or phase characteristics over a prescribed frequency range. Obviously, there will be certain restrictions on the assigned transmission function if the network is to be physically realizable. Further, the solution will not be unique, though certain solutions may be more convenient than others. Engineering and cost requirements usually impose severe limitations on the number of elements that may be used in constructing a physical network, hence it may not be possible to match the given function exactly even within the prescribed range of frequencies. Thus from the practical design point of view the problem of network synthesis may be formulated as follows: *To design a network with a reasonable number of lumped elements such that its transmission function approximates a given transmission function to a prescribed tolerance in a given frequency range.*

The potential analogue method of network synthesis is a method of approximating to the prescribed transmission function by considering charge distributions in a complex plane and their associated potential and stream functions. In other words, the fact that the prescribed function is usually analytic means that its real and imaginary parts are potential functions

which satisfy Laplace's equation in two dimensions. Hence they may be interpreted as potential and stream functions (interchangeably) of certain charge distributions. In potential theory the problem of network analysis corresponds to the problem of determining the potential of a given charge distribution, while the problem of network synthesis corresponds to the problem of determining an appropriate charge distribution when the potential is given.

This is one of the fundamental problems of potential theory, and it has been widely discussed in the mathematical literature of the subject. The usefulness of the potential analogue method of network synthesis derives primarily from the fact that we may use the whole background of our knowledge of potential theory and of the properties of electrostatic fields in formulating the solution of the charge distribution problem. A general solution is obtained in terms of a continuous distribution of charge over a contour (C) in the complex plane. This is the mathematical part of the problem. Thereafter, the design problem is to approximate the continuous distribution by means of a set of lumped charges which will have approximately the same potential function. The solution of this problem involves a certain amount of ingenuity, and may at times seem to be more of an art than a science. Once the lumped charge distribution has been determined, the locations of the charges are interpreted as corresponding locations of poles and zeros of the transmission function. Well-known methods of designing a network with assigned poles and zeros may then be used, and the problem regarded as solved.

We may note that neither the lumped charge distribution nor the contour (C) is uniquely determined by a given transmission function. Physical restrictions on the type of distribution which will lead to a realizable network usually impose sufficient limitations on the charge distribution, but the contour (C) remains to some extent at our disposal. If our first choice of contour proves unsatisfactory we can always try another contour which may give more suitable results. This introduces another important characteristic of the potential analogue method, namely that we may use the properties of conformal transformations to simplify the choice of contour. Thus any simple closed contour in the complex p-plane may be mapped on a unit circle in a second complex plane. The solution of the charge distribution problem on the unit circle is particularly simple, but it may not lead to the most suitable network design formula. However, we may use the inverse transformation to map the unit circle on some more convenient contour and locate equivalent charges at corresponding points of the two contours.

From the mathematical standpoint the use of continuous charge distribution instead of lumped charges corresponds to the use of integrals

instead of finite sums. To the best of the author's knowledge, the first application of the continuous charge concept to network synthesis was by H. W. Bode, who used the so-called "condenser plate" analogue to design phase equalizers for experimental coaxial cable systems for television, in the late nineteen-thirties. An extension of the "condenser plate" technique, combining gain and phase equalization, is described in a patent issued to Bode[1] in 1944. The integration idea was used independently by W. Cauer,[2] in connection with applications of Poisson's integrals to network problems. Development of the potential analogue method was interrupted by the war, but in the last few years there has been considerable activity in this field.[3] The aim of the present paper is to systematize the development of the potential analogue method, and to extend it in various directions in order to obtain a more versatile design tool. Much of the material has been presented orally at meetings of the Basic Science Division of the A.I.E.E.[4]

In principle, at least, the method may be used to simulate or equalize, over a finite range of useful frequencies, any gain or phase characteristic

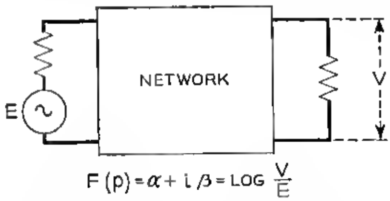

Fig. 1—A transducer used as a transmission circuit.

which may be represented by an analytic function. Network types to which the method has been successfully applied include filters, equalizers, delay networks and combinations of networks required for long communication systems such as coaxial cables. As experience increases, the range of applications is still being extended.

2. Analytic Properties of the Transmission Function

We shall consider the transmission function of a typical transducer, Fig. 1. The absolute value of the ratio of the output voltage to the input voltage represents the gain in transmission through the network, while the phase of the ratio represents the phase shift. If α is the gain in nepers and β the phase shift in radians we have

$$V/E = e^{\alpha}e^{i\beta}, \tag{1}$$

and we define the transmission function as the logarithm of this ratio,

$$F(i\omega) = \log (V/E) = \alpha + i\beta. \tag{2}$$

For a finite network with lumped elements the ratio V/E is a rational fraction and the transmission function may be represented by an expression of the form

$$
\begin{aligned}
F(p) &= \log K \frac{(p - p_1')(p - p_2')\cdots}{(p - p_1'')(p - p_2'')\cdots} \\
&= \log K + \sum \log (p - p_m') - \sum \log (p - p_n''),
\end{aligned} \tag{3}
$$

where K is a constant which may usually be ignored in the analysis since its value merely alters the *level* of gain or phase and does not affect their variation with frequency. We have introduced the complex oscillation constant

$$p = \xi + i\omega \tag{4}$$

instead of the real frequency variable, ω, and equation (3) defines the transmission function in the complex p-plane. If we separate the real and imaginary parts of (3) we find analytic expressions for the gain and phase:

$$
\begin{aligned}
\alpha &= \alpha_0 + \sum \log | p - p_m' | - \sum \log | p - p_n'' |, \\
\beta &= \beta_0 + \sum ph(p - p_m') - \sum ph(p - p_n'').
\end{aligned} \tag{5}
$$

The significance of the parameters p_m' and p_n'' is easily understood if we note that when $p = p_m'$ we have $\alpha = -\infty$ and therefore $V/E = 0$. Hence the zeros of the rational fraction in (3) represent points of infinite loss of the network. Similarly if $p = p_n''$ then $\alpha = \infty$ and we may have a finite value of V when E is zero. Thus the poles of the rational fraction are the natural oscillation constants or natural modes of the network. For brevity we shall refer to p_m' and p_n'' as the zeros and poles of $F(p)$ though they are really logarithmic singularities of the transmission function.

The numerator and denominator of the rational fraction are finite polynomials in p. If the network consists of real elements the coefficients in the polynomials are real. Thus we have the first property of the transmission function. The *zeros and poles must be either real or conjugate complex*. A second essential property is that *the real parts of the poles p_n'' must be negative* if the network is to be stable. And the third property that concerns us is that *there must be at least as many poles as zeros*, that is, as many finite natural modes as points of infinite loss. This condition insures the proper behavior of the transmission function at asymptotically high frequencies.

Using these properties the gain and phase may be expressed in alternative forms. From the first property it follows immediately that the conjugate function $[F(p)]^*$ must be equal to the value of F when $p = p^*$. But $p^* = -p$ when $p = i\omega$, hence in this case

$$[F(p)]^* = F(-p) = \alpha - i\beta. \tag{6}$$

On the real frequency axis, therefore, we have

$$\begin{aligned} \alpha &= \tfrac{1}{2}[F(p) + F(-p)] = \text{even part of F}, \\ i\beta &= \tfrac{1}{2}[F(p) - F(-p)] = \text{odd part of F}. \end{aligned} \tag{7}$$

Specifically we may write

$$\begin{aligned} 2\alpha &= 2\alpha_0 + \sum \log | p_m'^2 - p^2 | - \sum \log | p_n''^2 - p^2 |, \\ 2\beta &= 2\beta_0 + \sum ph\left(\frac{p_m' - p}{p_m' + p}\right) - \sum ph\left(\frac{p_n'' - p}{p_n'' + p}\right), \end{aligned} \tag{8}$$

where the singularities occur in pairs, one of each pair being the negative of the other.

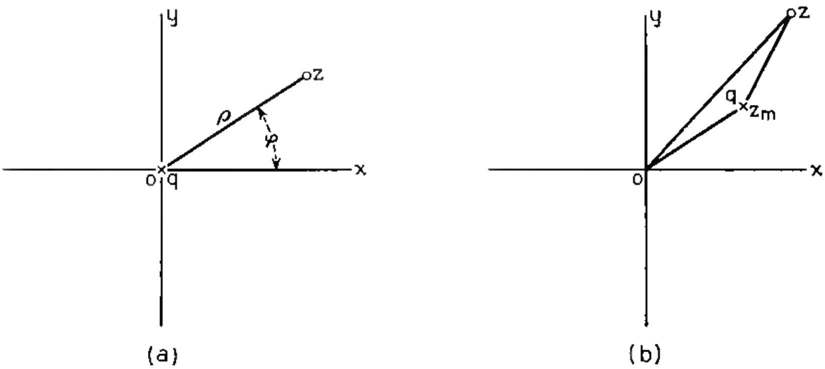

Fig. 2—A point charge in the potential plane; (a) at the origin, (b) at the point z_m.

3. Logarithmic Potentials

In two-dimensional potential theory we are really concerned with uniformly charged line filaments whose potentials and intensities are the same in any plane perpendicular to the axis of the filament. Hence, it is convenient to speak of a point charge q in a two-dimensional plane (x, y) and regard the plane as the plane of a complex variable, $z = x + iy$. The potential of a charge q at the origin in this plane, Fig. 2(a), is proportional to the magnitude of the charge and to the logarithm of the distance from the charge.

$$V = -q \log \rho + \text{constant}, \tag{9}$$

where the constant may have any convenient value. Note that we are using arbitrary units of charge and potential; in a coherent system of electromagnetic units the logarithmic term would have a constant multiplier.

For present purposes this would merely lead to a complication of the argument.

If we introduce polar coordinates, $z = \rho e^{i\varphi}$, we may consider a complex potential

$$W = -q \log z + \text{constant} = -q \log \rho - iq\varphi + \text{constant}. \quad (10)$$

The real part of this function is the potential and the imaginary part is the stream function. If the charge is at a point z_m, other than the origin, Fig. 2(b), the corresponding complex potential is

$$W = -q \log (z - z_m) + \text{constant}. \quad (11)$$

For a set of point charges the total potential is simply the sum of the individual potentials,

$$W = -\sum q_m \log (z - z_m) + \text{constant}, \quad (12)$$

while for a continuous distribution of charges over a contour (C) the sum is replaced by an integral,

$$W = -\int_{(C)} Q(\zeta) \log (z - \zeta) \mid d\zeta \mid, \quad (13)$$

where $\mid d\zeta \mid$ is an element of length on the contour.

In general we write

$$W = V + i\Psi \quad (14)$$

where V is the potential and Ψ the stream function. We note that W in (12) is analytic everywhere in the finite part of the z-plane except at points occupied by the charges. Similarly, W in (13) is analytic everywhere except on the contour (C) and at infinity.

We may use the theory of analytic functions of a complex variable to obtain various properties of the potential and of the stream function. First, we remark that the derivative of W is unique, and may be written in either of the forms

$$\frac{dW}{dz} = \frac{\partial V}{\partial x} + i\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial y} - i\frac{\partial V}{\partial y}, \quad (15)$$

whence V and Ψ satisfy the Cauchy-Riemann relations,

$$\frac{\partial \Psi}{\partial x} = -\frac{\partial V}{\partial y}, \qquad \frac{\partial \Psi}{\partial y} = \frac{\partial V}{\partial x}. \quad (16)$$

The stream function and the potential are not independent; either is determined by the other except for a constant.

The components of the electric intensity are obtained from V by the relation $E = -\text{grad}\, V$. Thus we find various alternative forms,

$$E_x = -\frac{\partial V}{\partial x} = -\frac{\partial \Psi}{\partial y} = -\operatorname{re}\frac{dW}{dz},$$
$$E_y = -\frac{\partial V}{\partial y} = \frac{\partial \Psi}{\partial x} = \operatorname{im}\frac{dW}{dz}. \tag{17}$$

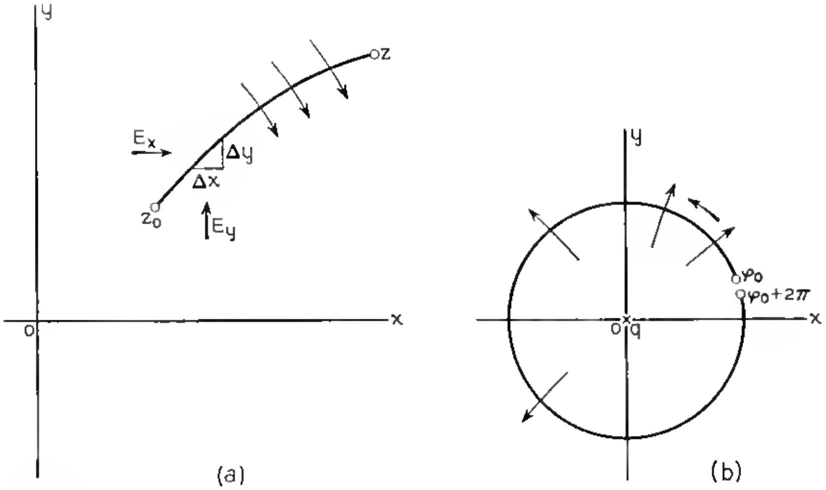

Fig. 3—Flux of electric intensity; (a) across an arc, (b) through a circle surrounding a charge.

The stream function Ψ may be interpreted in terms of the flux of the field intensity across a curve in the z-plane. The flux of a vector across a given curve is the line integral of the normal component of the vector,

$$\Phi = \int E_n \, ds, \tag{18}$$

hence the flux of E crossing the curve of Fig. 3(a) between the points z_0 and z is

$$\Phi = \int_{z_0}^{z} (-E_y \, dx + E_x \, dy)$$
$$= \int_{z_0}^{z} \left(-\frac{\partial \Psi}{\partial x} dx - \frac{\partial \Psi}{\partial y} dy \right) = \Psi(z_0) - \Psi(z), \tag{19}$$

in the clockwise direction when viewed from z_0. The flux depends only on the values of Ψ at the ends of the curve.

For a point charge q at the origin the stream function is

$$\Psi = -q\varphi + \text{constant}, \tag{20}$$

and the outward flux through a closed contour surrounding the charge, Fig. 3b, is

$$\Phi = -q\varphi_0 + q(\varphi_0 + 2\pi) = 2\pi q. \tag{21}$$

The flux from a set of charges is additive, so that equation (21) is general, when q is interpreted as the total charge inside the contour.

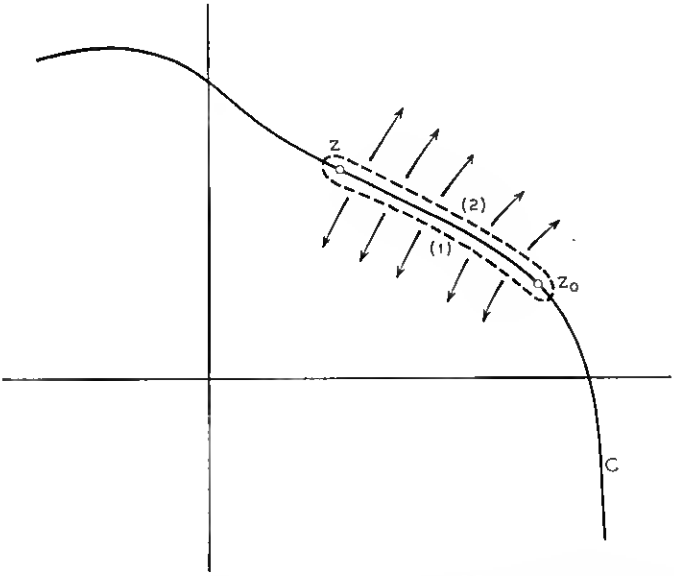

Fig. 4—Narrow closed contour surrounding an arc whose accumulated charge is $q(z)$.

Consider now a charge distributed continuously on a contour (C), and let $q(z)$ be the total charge on the arc extending from z_0 to z. If we surround this arc by an infinitely narrow closed contour, Fig. 4, we can pass from z_0 to z on the enclosing contour in either a clockwise or a counterclockwise manner, by traversing respectively part 1 or 2 of the contour, on one or the other side of the enclosed arc of C. The flux leaving the enclosing contour through part 2 is

$$\Phi_2 = \Psi_2(z_0) - \Psi_2(z), \tag{22}$$

where Ψ_2 is the stream function in the region on the corresponding side of (C). Similarly the flux leaving the enclosing contour through part 1 is

$$\Phi_1 = \Psi_1(z) - \Psi_1(z_0), \tag{23}$$

where Ψ_1 is the stream function in the region on that side of (C). Since the total flux, $\Phi_1 + \Phi_2$, is given by (21) we see that the stream function is discontinuous across the line charge, and the amount of the discontinuity is

$$[\Psi_1(z) - \Psi_1(z_0)] - [\Psi_2(z) - \Psi_2(z_0)] = 2\pi q(z). \tag{24}$$

If C is a closed contour, and if the above arc corresponds to passage from z_0 to z in a *counterclockwise* direction around C, then Ψ_1 and Ψ_2 in (24) correspond respectively to the interior and exterior of C.

On the other hand the potential is continuous across the line cbarge. To prove this we note that the potential is the real part of the complex potential W in (13), and is therefore given by

$$V = -\int_{(C)} Q(\zeta) \log |z - \zeta| |d\zeta| + \text{constant.} \tag{25}$$

The integral depends on the distance $|z - \zeta|$ between a typical point ζ on (C) and tbe given point z. For two points z_1 and z_2 just on opposite sides of (C) the distance is the same, so tbat $V(z_2) = V(z_1)$.

4. Analogy Between Transmission Functions and Logarithmic Potentials

Comparing equations (3) and (12) we see that tbe transmission function $F(p)$ in the complex p-plane may be identified with the complex potential W of a system of discrete cbarges. If we assume that unit positive charges are located at the natural modes, p_n'', of the network, and unit negative charges at the infinite loss points, p_m', the complex potential in the p-plane is

$$W = -\sum \log (p - p_n'') + \sum \log (p - p_m') + \text{constant.} \tag{26}$$

The real part of this function is the potential and its imaginary part is the stream function. Then, by the definition of gain and phase in equation (2), the gain of the associated network is given by the potential on the imaginary axis (the real frequency axis), and the phase hy the corresponding stream function.

The zeros and poles of $F(p)$ locate the charges producing the complex potential W, and they form a discrete set of points. When $F(p)$ corresponds to practical problems these points are usually arranged along well-defined lines in the complex p-plane and not distrihuted at random throughout a whole area. The corresponding potential W should then be that of a discrete set of charges arranged along corresponding lines in the charge plane. When the potential function is given in analytic form, however, it is usually simpler to use known methods of potential theory to determine a continuous charge distrihution over a convenient contour. This continuous distrihution may then be approximated by a set of *equal* lumped units of charge spaced on the same contour. The difference between the actual 'sources' of $F(p)$ and W is usually small, and by using distributed charges much of the algebraic complexity associated with the design of complicated networks may be avoided, at least in the earlier stages.

When the assigned gain or phase is represented in analytic form it is sometimes possible to determine a distributed charge over a suitably chosen contour which matches the desired characteristic exactly. Then the only approximations involved in obtaining a finite network are those which arise from replacing the continuous charge distribution by a set of lumped charges. The errors are easy to calculate and can usually be adjusted to meet the allowable network tolerance.

It is important to stress that for physical networks the complex potential W must be generated by unit charges. Hence, if we have determined a continuous charge distribution over a given contour in the complex p-plane, we must choose our unit of charge to make the total charge on the contour equal to an integral number of charge units. Then the contour can be divided into segments each carrying a unit charge, and the lumped charge distribution is obtained by locating one unit of charge at some convenient point on each segment, usually at or near the center. The total charge determines the number of lumped charges that may be used. This limitation is not so restrictive as it might appear at first sight, since the assigned transmission function frequently involves a constant parameter in terms of which the unit of charge may be defined. It is also possible, as we shall see later, to increase the total charge on the contour by special devices, appropriate to different types of problem.

We assume that the gain, α, corresponds to the real potential, V, and the phase, β, to the stream function Ψ; but it would be equally permissible to interpret α as the stream function of another complex potential, iW, and then β would be the negative of the potential. It is usually more convenient to equate gain and potential, in network synthesis problems, and we shall confine our analysis to this interpretation.

The desired form of gain and phase may be given as a condition on their variation with frequency. Since the electric intensity is the gradient of the potential, we see from equations (17) that $d\alpha/d\omega$ is analogous to the electric intensity in the direction of the negative frequency axis. Similarly, the variation of β with frequency is analogous to the electric intensity in the direction of the negative real p-axis, that is, at right angles to the frequency axis. Thus we may summarize the analogies we shall use most frequently:

a) Transmission function and complex potential
b) Gain and potential
c) Phase and stream function
d) $-\dfrac{d\alpha}{d\omega}$ and field along real frequency axis (27)
e) $-\dfrac{d\beta}{d\omega}$ and field across real frequency axis.

The conditions imposed on the zeros and the poles of the transmission function to make it physically realizable have their counterparts which must be imposed on the charge distribution associated with the complex potential if it is to be equivalent to a realizable network. Using the above analogies they may be summarized as follows:

1) The charge distribution must be symmetrical about the real axis in the complex plane.
2) The positive charges must be in the negative half of the plane.
3) The net charge must be non-negative.
4) If the contour is made up of disjoint curves in the plane there must be an integral number of units of charge on each segment.

(28)

The first three conditions correspond exactly to the zero and pole limitations, while the last is a corollary of the unit charge limitation we have already discussed.

5. Condenser Delay Networks

As a simple example of the potential analogy we shall consider the design of a network with constant phase delay in a prescribed frequency range. Analytically the condition is that $d\beta/d\omega$ should be constant for $|\omega| < \omega_0$, where ω_0 has an assigned value. The corresponding function in the potential plane is the field transverse to the imaginary axis. This suggests the field between the plates of a parallel plate condenser, and we construct immediately the analogy illustrated in Fig. 5. The distributed charge is shown in Fig. 5a, where we assume a constant charge density on each plate of the condenser, the plates being parallel to the real frequency axis. The positive charge is placed on the left-hand plate to satisfy the second condition of the set (28).

As long as the distance between the plates is small compared with their width the field between the plates is transverse, and substantially constant, except for an edge effect which will diminish as the dimensions of the plates are increased. If we could use infinite plates the field would be exactly constant, and the continuous charge distribution on the plates would match the network stipulation exactly. In practice we must use a finite number of lumped charges; hence we choose the charge points shown in Fig. 5b, where the crosses represent unit positive charges, the natural modes of the network; and the circles represent unit negative charges, the infinite loss points. To keep the end effects small it is desirable to extend the plates considerably beyond the frequency ω_0.

We note that for the lumped charge distribution the field along the real frequency axis vanishes, since each unit positive charge contribution is

cancelled by the contribution from the opposite negative charge. Thus, by our analogy, $d\alpha/d\omega$ vanishes, and the constant phase delay network has a constant gain at all frequencies.

For the charge spacing illustrated in Fig. 5 the poles of the transmission function are located at the positive charge points, $p''_\mu = -a + i\nu b$, $\nu = -m, \cdots 0, \cdots m$, while the infinite loss points are located at the negative charge points, $p'_\mu = +a + i\nu b$. Thus the required transmission function is

$$F(p) = \text{constant} + \log \prod_{\mu=-m}^{m} \frac{p - a - i\mu b}{p + a - i\mu b}. \tag{29}$$

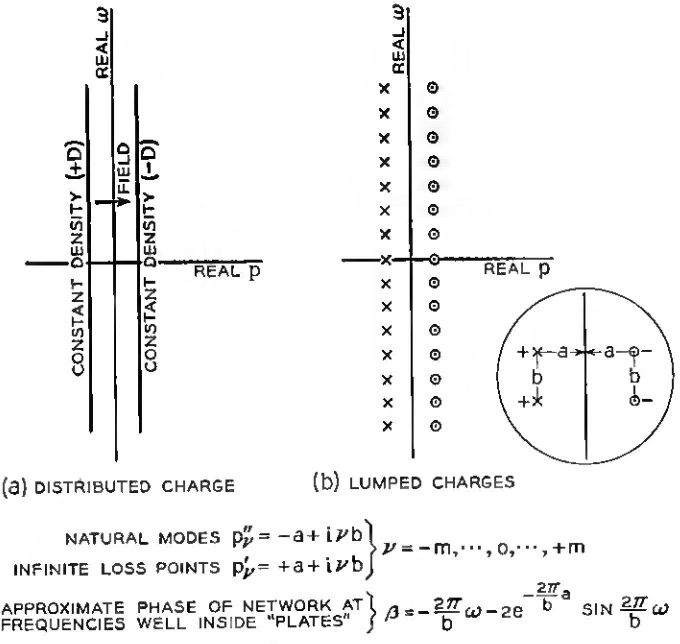

Fig. 5—The condenser plate analogue; (a) distributed charge, (b) lumped charges.

If we allow the number of charges to become infinite, but still with constant spacing b, the infinite product may be recognized as the ratio of sine or cosine functions,*

$$F = \text{constant} + \log \frac{\sin, \cos\left(\dfrac{p - a}{ib}\pi\right)}{\sin, \cos\left(\dfrac{p + a}{ib}\pi\right)}, \tag{30}$$

* See, for instance, B. O. Pierce's "Short Table of Integrals," page 96, equations 816, 817.

where the sine or cosine is used according as the number of oscillation constants is odd or even.

There are two sources of error in the finite representation (29): The first is due to the finite extent of the charged plates, and may be called the "truncation error." Its effect will be important only near the ends of the plates, which explains why it is advisable to prolong the charges beyond the upper frequency bound ω_0. Its magnitude is exactly determined by integrating the effect of uniform charge density, of magnitude $1/b$, over the region beyond the finite plates:

$$-\frac{d\beta}{d\omega} = \frac{2\pi}{b} - \left[\frac{2}{b}\tan^{-1}\frac{a}{\omega_e + \omega} + \frac{2}{b}\tan^{-1}\frac{a}{\omega_e - \omega}\right], \tag{31}$$

where $\pm\omega_e$ are the real frequencies at the ends of the plates. The bracketed expression represents the non-constant part of the phase delay, due to the finite extent of the plates. Note that $2\omega_e = nb =$ total extent of natural mode intervals = plate width. The correction term becomes smaller as ω_e increases.

The second source of error lies in the use of lumped charges instead of a continuous charge distribution, and may be called the "granularity error." Its magnitude may be approximately determined from (30) if we replace the sines and cosines by their exponential equivalents, differentiate with respect to ω, and assume that the error is small. We find.

$$-\frac{d\beta}{d\omega} = \frac{2\pi}{b} \pm \frac{4\pi}{b}\exp\left(-\frac{2\pi a}{b}\right)\cos\frac{2\pi\omega}{b}, \tag{32}$$

where the plus and minus signs refer respectively to odd and even numbers of modes.

We may assume that both errors are small, and that they act independently, so that the total error is given approximately by the sum of the non-constant factors in (31) and (32). We note that if we increase the plate spacing, a, the granularity error becomes smaller while the truncation error increases. This increase may be offset by increasing ω_e, but this means extending the condenser plates and therefore adding additional lumped charges, with consequent increase in network complexity. Thus the choice of specific spacing and dimensions is likely to represent a balance between granularity errors, truncation errors and network complexity.

The truncation errors may be somewhat reduced, with no increase in network complexity, by increasing the charge densities near the edges of the plates. Later we shall discuss a systematic method of adjusting the charge distribution.

6. Filters or Selective Networks

Filters offer another particularly simple illustration of the potential analogy. The object of a filter is to transmit all frequencies in a prescribed range and to block all other frequencies. This means that the potential must be substantially constant in the pass-band, and large and negative in the stop-band. Now the potential inside a conductor *is* constant, hence charge distributions on conductors should yield transmission functions of filters.

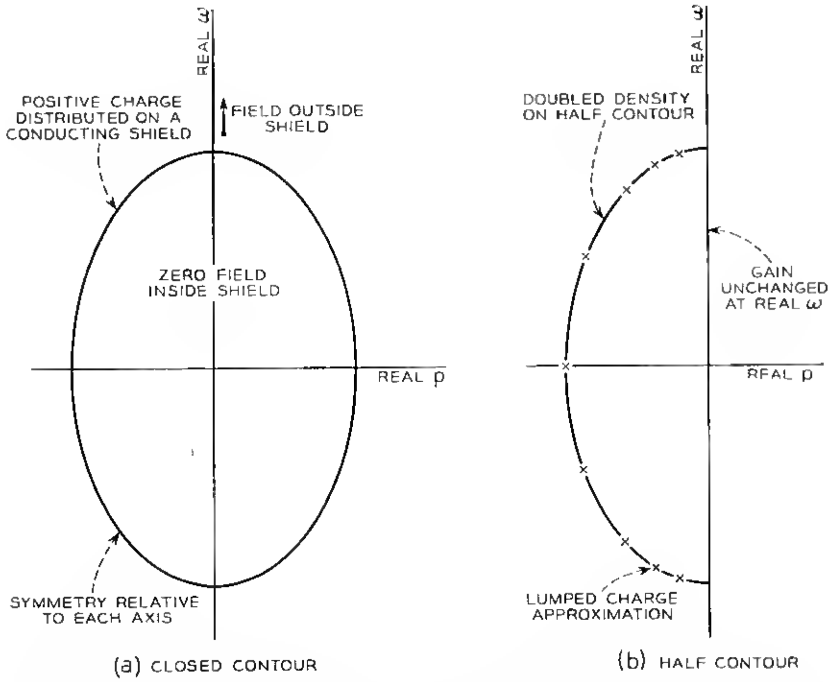

Fig. 6—Analogy between filters and conducting shields; (a) positive charge distributed on symmetric shield, (b) lumped charge distribution on half of contour.

Figure 6 illustrates the analogy between filters and conductors, or shields. The first condition in the set (28) requires that the shield must be symmetric in the real p-axis. Symmetry about the ω-axis is not necessary, but it is usually advantageous. The third condition of (28) requires that the charge on the shield should be positive, in the absence of external charges. Positive charges determine the poles of $F(p)$, and must therefore lie in the left half of the p-plane if the network is to be physically realizable. In the shield, on the other hand, there are positive charges in both halves of the p-plane, so that we cannot use the charge distribution on the shield without modi-

fication. The difficulty is readily resolved, however, if we note that the charge on the shield is symmetric about the ω-axis, and that the charges on each half of the shield produce the same potential *on* the imaginary axis. Hence the gain will be unchanged, if we use only the left half of the shield and double the charge.

Even if the shield is not symmetrical about the ω-axis we can still transfer the positive charges on the right half of the plane to their mirror images in the axis without changing the value of the potential *on* the axis. This

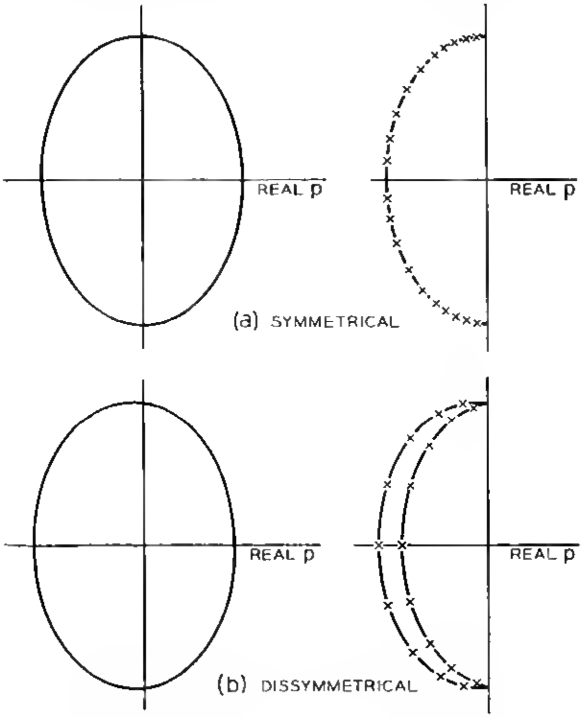

Fig. 7—Lumped charge distribution for a given contour; (a) symmetrical, (b) dissymmetrical.

would give us a charge distribution over two separate contour branches, as in Fig. 7, and would thus increase the network complexity. This explains the desirability of using the type of shield which is symmetrical relative to each axis.

So far we have considered conductors in the absence of external charges (except at infinity). If the network is to have points of infinite loss at certain finite frequencies we must have negative charges outside the shield, Fig. 8. These charges alter the charge distribution on the shield, but the potential

inside the shield is still constant. In the case of band-pass filters we can use disjoint contours as in Fig. 9. These must be symmetric about the ξ-axis and again we shall find it advantageous to have them symmetric also about the ω-axis. In all cases the net charge on the shield must be positive, and

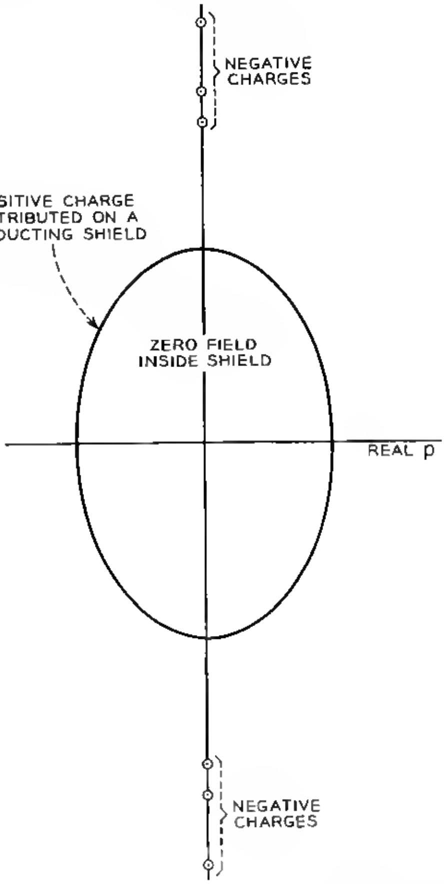

Fig. 8—Negative charges outside shield have no effect on potential inside.

we can state as a general filter principle that: *The natural oscillation constants "shield" pass-bands from infinite loss points.*

7. Gain Invariant and Phase Invariant Transformations

We have just seen that we can transfer positive charges (or poles) from the right half of the p-plane to the left without changing the value of the potential (or gain) *on* the real frequency axis. Similarly, there are trans-

formations which leave the stream function (or phase) unaltered. These invariant transformations are easy to understand if we consider the components of the field intensity. As shown in Fig. 10a the field of any given charge along the ω-axis equals the field of an equal charge at the mirror image of the given charge in the real frequency axis. By (27d) these two charges thus give the same rate of change of α with frequency. Similarly two opposite charges, Fig. 10b, at mirror image points have the same

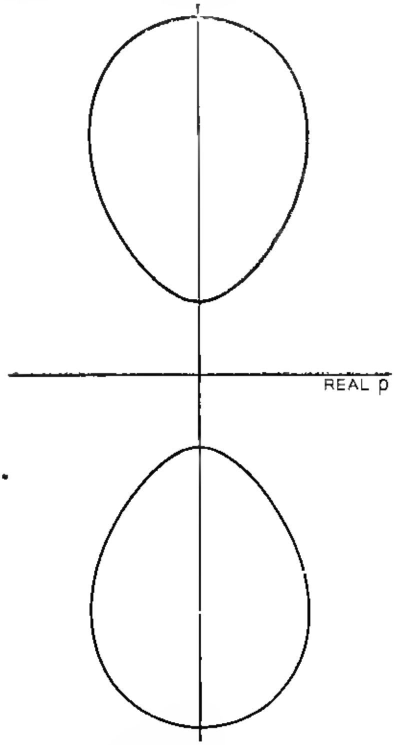

Fig. 9—A disjoint contour.

transverse field intensity *across* the ω-axis. Thus these charges produce the same rate of change of β with frequency.

To summarize in terms of the transmission functions: 1) the zeros and poles of $F(p)$ may be moved from the right half of the p-plane to the left half, and vice versa, without changing tbe gain; 2) a singularity of $F(p)$ may be moved from one half of the p-plane to the other without changing the phase, provided the type of singularity is reversed (that is, a zero becomes a pole and vice versa).

8. Green's Formula

The simple examples we have just discussed could bave been solved without recourse to the potential analogous method, since the charge distributions were easy to recognize. In general, this is not the case, and we now turn to systematic methods of determining the charge distribution when the gain, α, is given as an analytic function of ω in a prescribed frequency range, $|\omega| < \omega_0$. The corresponding transmission function is obtained if we replace ω by p/i and regard p as a complex variable. Then the mathematical problem is to determine a charge distribution on some contour C which will have this function $F(p)$ as its complex potential.

The contour C is to a large extent arbitrary. We shall assume that it is a simple closed curve in the p-plane, enclosing the frequency band of interest, and subject only to the limitations that it must be symmetric in the real p-axis, and that $F(p)$ must be analytic inside C.

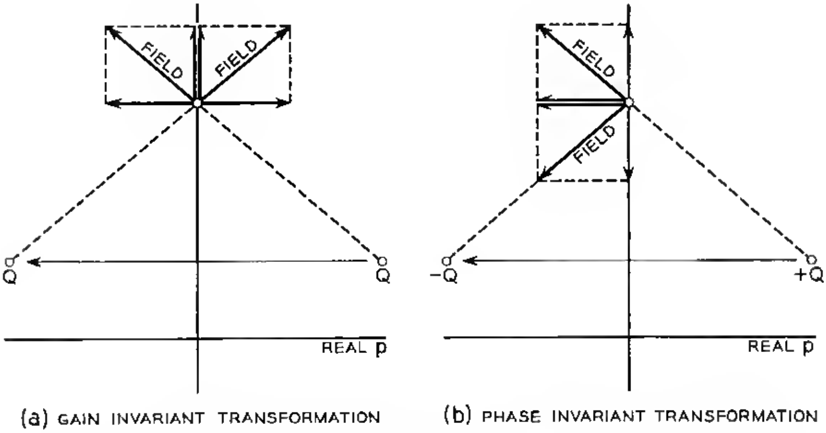

Fig. 10—Illustrations for (a) gain invariant (b) phase invariant transformations.

Then one very general solution of the charge distribution problem in potential theory is given by Green's formula,† which has the form

$$V(P) = \frac{1}{2\pi} \int_C \left(V \frac{\partial}{\partial n} \log \rho - \frac{\partial V}{\partial n} \log \rho \right) ds \tag{33}$$

for the logarithmic potential in two dimensions. The integral expresses the potential at any point P inside C in terms of the values of V and of its normal derivative *on* C. The differential ds is an element of length on C and n is the normal drawn out of the region we are considering. At points

† See e.g. A. G. Webster, Partial Differential Equations of Mathematical Physics, G. E. Stechert and Co., New York, 1927, p. 210.

outside C the integral vanishes. In potential theory it is shown that the potential V on C may be interpreted as a double layer of charge of strength V, while the normal derivative of the potential on C may be interpreted as a single layer of charge of density $\partial V/\partial n$. Thus Green's formula expresses the potential inside C as due to a single *and* double layer of charge on C, the charges being determined by the known values of V inside C.

Green's formula represents a very simple and general solution of the charge distribution problem. The simplicity is due primarily to the constancy of the potential outside C; and this in turn is made possible by the double layer of charge, which supplies the discontinuity between the variable interior and constant exterior potentials. Unfortunately, from the network synthesis point of view, it is not a practical solution, for double layers of charge lead to zero-pole combinations which are not easily realizable. A double layer might be approximated by two closely-spaced strings of positive and negative charges, but the resulting zeros and poles would be in addition to the zeros and poles for the simple layer of charge. Hence the associated network would be difficult to design, and would also be unnecessarily complicated and wasteful of network elements.

It is well-known, however, that V and its normal derivative cannot both be assigned independently on C, and that the potential inside C is determined when we know the values of V alone on C. This would make it possible to eliminate the double layer of charge, if we could obtain the analytic continuation of V on both sides of C. Then we should have a potential which is continuous across C, and this would be consistent with the existence of a simple layer of charge on C whose density is determined by the discontinuity in the associated stream function, as we saw in Section 3.

We might remark that if $V(P)$ is a given function of P *outside* C, the integral (33) will again express $V(P)$ at points outside C in terms of the values of V and $\partial V/\partial n$ on C. In this case $V(P)$ must vanish at infinity at least as $1/\rho$, and the value of the integral will be zero at all points inside C. Hence if we retain both single and double layers of charge it is possible to obtain a charge distribution on C for which the gain characteristics are assigned over the *entire* frequency axis. With simple layers the gain may be assigned only over that part of the frequency axis which lies inside C. Then we must accept its values on the remainder of the axis, though it may be possible to control these values to some extent by varying the contour C.

9. The Exterior Transmission Function

We have just seen that for a simple layer of charge on C we have to determine the analytic continuation of the transmission function on both

sides of C. Then the potential will be continuous across C while the stream function will be discontinuous by an amount which is determined by the charge on C in accordance with equation (24). If we write

$$F_i(p) = V_i(p) + i\Psi_i(p) \tag{34}$$

for our known function inside C, and a corresponding expression

$$F_e(p) = V_e(p) + i\Psi_e(p) \tag{35}$$

for the complex potential outside C, then V_e is determined by V_i, while Ψ_e will he known if we know both Ψ_i and q. Conversely, q will be determined if we know both Ψ_i and Ψ_e. Thus the prohlem of determining the charge distribution on C may also be formulated as the problem of determining the exterior stream function Ψ_e. To make the function Ψ_e unique we specify that it must be analytic outside C, and must vanish at infinity at least as $1/p$, except perhaps for a logarithmic term which corresponds to an equipotential charge density on C. If the net charge on C is zero Ψ_e must vanish at infinity.

Thus, if it is possible to solve this potential prohlem we have a corresponding solution of the charge distribution problem. The *existence* of a solution has heen proved, and is known as Dirichlet's principle, but its solution has been formulated analytically only for circular contours. However, for circular contours in the p-plane simple methods of determining Ψ_e are available, and we shall discuss these before giving the general solution.

10. The Power Series Solution for a Circular Contour

When the interior transmission function is given as an analytic function inside a circular contour, the exterior function may be determined by various methods. An elementary method is based on power series expansions. Since any analytic function of p can be expanded in a power series inside a certain domain of convergence the method has quite general application. To obtain the best form of power series applicable to our problem, we shall start by considering the expansion of the complex potential for a given set of lumped charges q_n located on the circle at points p_n,

$$F(p) = \text{constant} - \sum_n q_n \log (p - p_n). \tag{36}$$

Inside the circle we have $| p | < p_n$ for each of the charge points p_n, and therefore each of the logarithmic terms may be expanded as convergent series in p/p_n. Hence

$$\begin{aligned} F_i(p) &= \text{constant} - \sum_n q_n \log (-p_n) - \sum_n q_n \log \left(1 - \frac{p}{p_n}\right) \\ &= \text{constant} - \sum_n q_n \left[- \frac{p}{p_n} - \frac{p^2}{2p_n^2} - \cdots \right], \end{aligned}$$

and for the *interior* potential a suitable power series expansion is

$$F_i(p) = a_0 + \sum_{m=1}^{\infty} a_m p^m. \tag{37}$$

Outside the circle we have $|p| > p_n$, so that the logarithmic terms may be expanded in convergent series of p_n/p,

$$\begin{aligned} F_e(p) &= \text{constant} - \sum_n q_n \log p - \sum_n q_n \log\left(1 - \frac{p_n}{p}\right) \\ &= b_0' - b_0 \log p - \sum_n q_n \left(-\frac{p_n}{p} - \frac{p_n^2}{2p^2} - \cdots\right). \end{aligned}$$

Hence a suitable power series expansion for the exterior potential is

$$F_e(p) = b_0' - b_0 \log p + \sum_{m=1}^{\infty} b_m p^{-m}. \tag{38}$$

The constant b_0 represents the total charge on the circle. If there is no net charge the logarithmic term vanishes and $F_e(p)$ is analytic outside C. It will vanish at infinity if we also have $b_0' = 0$, but for the moment we shall retain both constants, and apply the boundary conditions on C to determine the unknown constants b_m from the known constants a_m.

On the circle of radius ω_0 we have

$$p = \omega_0 e^{i\vartheta}, \tag{39}$$

so that just inside C the interior potential is

$$F_i(\vartheta) = a_0 + \sum_{m=1}^{\infty} a_m \omega_0^m e^{im\vartheta}, \tag{40}$$

while just outside C the exterior potential is

$$F_e(\vartheta) = b_0' - b_0 \log(\omega_0 e^{i\vartheta}) + \sum_{m=1}^{\infty} b_m \omega_0^{-m} e^{-im\vartheta}. \tag{41}$$

In our applications the constants a and b are real, hence we may separate the real and imaginary parts of (40) and (41), and find

$$V_i(\vartheta) = a_0 + \sum a_m \omega_0^m \cos m\vartheta, \qquad \Psi_i(\vartheta) = \sum a_m \omega_0^m \sin m\vartheta, \tag{42}$$

$$\begin{aligned} V_e(\vartheta) &= b_0' - b_0 \log|\omega_0| + \sum b_m \omega_0^{-m} \cos m\vartheta, \\ \Psi_e(\vartheta) &= -b_0\vartheta - \sum b_m \omega_0^{-m} \sin m\vartheta. \end{aligned} \tag{43}$$

The condition that V must be continuous across C determines the b's:

$$b_0' - b_0 \log|\omega_0| = a_0, \qquad b_m = \omega_0^{2m} a_m, \qquad m > 0. \tag{44}$$

Then the charge distribution is determined by the discontinuity in Ψ across C. If we measure the charge from the real axis, $\vartheta = 0$, we find from equation (24),

$$\begin{aligned} 2\pi q(\vartheta) &= \sum a_m \omega_0^m \sin m\vartheta - [-b_0\vartheta - \sum b_m \omega_0^{-m} \sin m\vartheta] \\ &= b_0\vartheta + \sum (a_m\omega_0^m + b_m\omega_0^{-m}) \sin m\vartheta. \end{aligned}$$

Hence we have two alternative formulations for q:

$$\begin{aligned} q(\vartheta) &= \frac{b_0\,\vartheta}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty} b_m\,\omega_0^{-m} \sin m\vartheta \\ &= -\frac{a_0\,\vartheta}{2\pi \log|\omega_0/\omega_0'|} + \frac{1}{\pi}\sum_{m=1}^{\infty} a_m\,\omega_0^m \sin m\vartheta\,. \end{aligned} \tag{45}$$

We have substituted $b_0' = b_0 \log|\omega_0'|$ for the constant b_0', where ω_0' is an undetermined frequency. The total charge on the contour is

$$q(2\pi) = b_0 = -a_0/\log|\omega_0/\omega_0'|\,. \tag{46}$$

Since the total charge must be non-negative this implies that $b_0 \geqq 0$, but a_0 may be either positive or negative, according as ω_0' is greater or less than ω_0.

The gain and phase are determined by the values of F on the real frequency axis, $p = i\omega$, hence, inside C,

$$\alpha_i = a_0 + \sum_{n=1}^{\infty} (-)^n\, a_{2n}\,\omega^{2n}, \qquad \beta_i = \sum_{n=0}^{\infty} (-)^n\, a_{2n+1}\,\omega^{2n+1}, \tag{47}$$

and outside C,

$$\begin{aligned} \alpha_e &= -b_0 \log|\omega/\omega_0'| + \sum_{n=1}^{\infty} (-)^n\, b_{2n}\,\omega^{-2n} \\ \beta_e &= \pm b_0\,\frac{\pi}{2} - \sum_{n=0}^{\infty} (-)^n\, b_{2n+1}\,\omega^{-2n-1}, \end{aligned} \tag{48}$$

where the minus sign in β_e refers to points on the positive half of the ω-axis, and the plus sign to points on the negative half.

We note that α is an even function of ω while β is an odd function. This agrees with equation (7) and it means that if only the gain is prescribed we know directly only the even coefficients, a_{2n}, in the power series expansion, of F_i. Hence we know only the even part of $F_i(p)$. But we have seen that the singularities in the logarithmic expression for α occur in pairs, one of each pair being the negative of the other. To determine the complete transmission function $F_i(p)$ we must assign one of each pair of singularities to $F(p)$ and the other $F(-p)$ in such a way that equations (7) and (8) are satisfied.

For the *unit* circle the exterior potential and charge equations take very simple forms. Corresponding to the interior potential

$$F_i(p) = a_0 + \sum a_m p^m, \tag{49}$$

we find

$$\begin{aligned} F_e(p) &= -Q \log (p/\omega_0') + \sum a_m p^{-m}, \\ q(\vartheta) &= \frac{Q\vartheta}{2\pi} + \frac{1}{\pi} \sum a_m \sin m\vartheta, \end{aligned} \tag{50}$$

where Q is the total charge on the unit circle, $Q = a_0/\log |\omega_0'|$. The coefficients in all three series are identical.

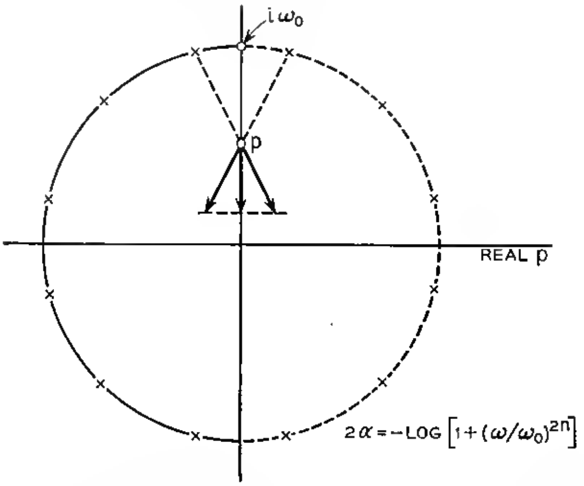

Fig. 11—Unit charges arranged symmetrically on a circle for the maximally flat filte approximation.

As a simple example let us determine the charge distribution on the unit circle which corresponds to a constant gain for $|\omega| < \omega_0$, and to a phase shift independent of ω. By equation (49) this requires that $a_m = 0$ when $m \neq 0$, and hence the continuous charge distribution on the circle is simply

$$q(\vartheta) = \frac{Q\vartheta}{2\pi}, \tag{51}$$

where Q is the total charge on the circle. Equal increments in ϑ give equal increments in the accumulated charge round the circle. If we ignore the requirements of realizability this distributed charge may be approximated by simply dividing the unit circle into $2m$ equal parts, and placing a unit positive charge at each point of division, Fig. 11,

$$p_k = e^{ik\pi/m}, \qquad k = 0, 1, \cdots 2m - 1. \tag{52}$$

The total charge is $Q = 2m$, and the transmission function for the lumped charge distribution is

$$F_i(p) = \text{constant} - \sum_{k=0}^{2m-1} \log(p - p_k). \tag{53}$$

Since

$$(p - p_0)(p - p_1) \cdots (p - p_{2m-1}) = p^{2m} - 1, \tag{54}$$

this is equivalent to

$$F_i(p) = \text{constant} - \log(p^{2m} - 1), \tag{55}$$

at all points inside the circle $|p| = 1$. This is the transmission function for the Butterworth "maximally-flat" filter.[5] As m increases F_i is more and more nearly constant inside C. But the objection to this solution is that it involves poles (or positive charges) in the right half of the p-plane. If the phase is of no importance we may use the gain invariant transformation to transfer these poles to the left half of the plane, which is equivalent to using only the left half of the contour, and doubling the charge at each charge point. Then we have a physically realizable charge distribution such that

$$q(\vartheta) = \frac{Q}{\pi}\left(\vartheta - \frac{\pi}{2}\right), \qquad \frac{\pi}{2} \leqq \vartheta \leqq \frac{3\pi}{2}. \tag{56}$$

For integral values of Q we locate charge points at $p_k = e^{i\vartheta_k}$, where

$$\vartheta_0 = \frac{\pi}{2} - \frac{\pi}{2Q}, \qquad \vartheta_{k+1} = \vartheta_k + \frac{\pi}{Q}. \tag{57}$$

The shape of the gain characteristic for small values of $Q = 2m$ is illustrated in Fig. 12. It approximates zero gain at frequencies inside the circle, and the approximation improves as m increases, or as the frequency decreases. At frequencies outside the circle the gain becomes a high loss, and the filter is of the low-pass type.

The transfer of poles from the right to the left half of the p-plane leaves the gain unaltered, but it changes the phase delay, since the sign of the phase contribution from each transferred charge is reversed. It is possible to compensate for this change by adding a simple charge distribution such as that shown in Fig. 13. Here the positive charges on the left are matched by the negative charges on the right, so that the electrostatic field is zero along the real frequency axis and the charges merely add a constant gain. The contribution to the phase delay from each negative charge equals that from the corresponding positive charge. Just as in the condenser plate analogue

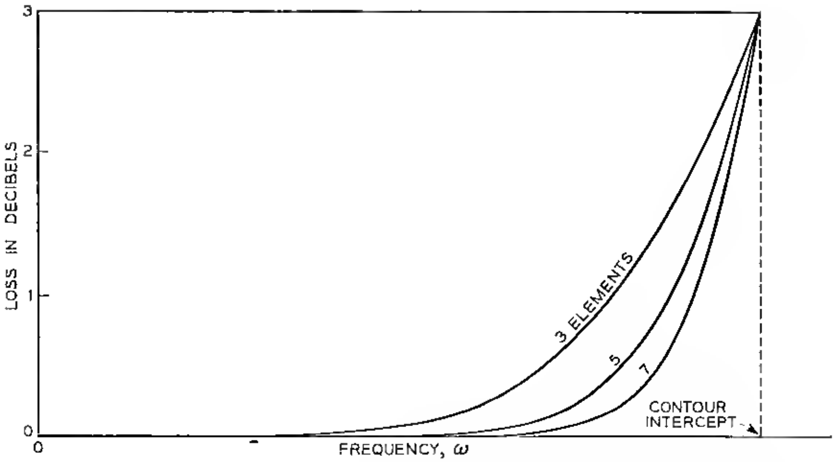

Fig. 12—Curves showing successive approximations to zero gain with maximally flat filters.

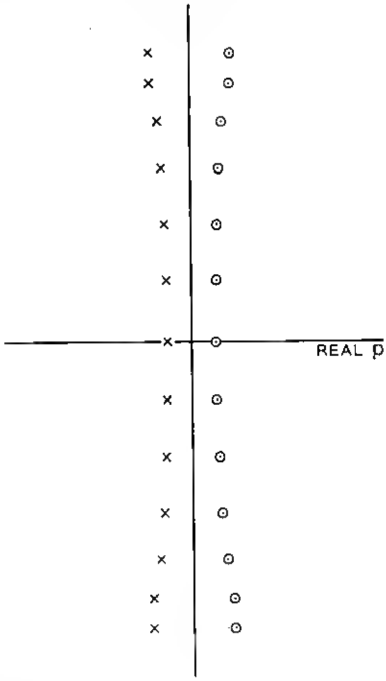

Fig. 13—A symmetrical charge distribution about the real frequency axis used to correct the phase delay.

of Fig. 5 the two sets of charges can be interpreted as equal and opposite charge densities on the two halves of the contour, thus giving a constant

phase delay. This method is of general application in changing the phase delay, and the corresponding networks are easy to obtain.

11. The Inversion Theorem for a Circular Contour

An alternative derivation of the exterior stream function for a circular contour of radius ω_0 is based on the method of inversion, in which p is replaced by ω_0^2/p, Fig. 14. This transformation maps the region inside C on the region outside C and vice versa. Points *on* the circle remain on the circle but are transformed to the conjugate complex points.

Now suppose that the transmission function $F_i(p)$ is defined inside the circle as an analytic function of p, and that it satisfies the conditions for physical realizability. Then if we have a unit charge at some complex point,

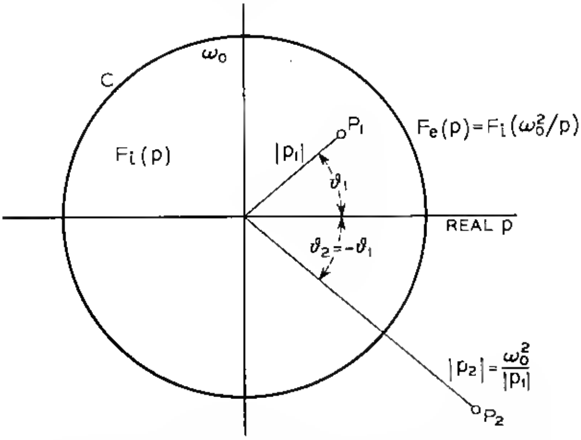

Fig. 14—The inversion theorem for a circular contour.

p on the circle there must be a like charge at the conjugate complex point, p^*, while the total charge must be non-negative. For simplicity we may assume that the total charge is zero and then the exterior function $F_e(p)$ must be analytic outside C. We wish to show that $F_i(\omega_0^2/p)$ may be interpreted as the exterior function. Obviously since $F_i(p)$ is analytic inside C we must have $F_i(\omega_0^2/p)$ analytic outside C. Hence it will represent the exterior function outside C if it represents a function whose potential is the analytic continuation of the potential inside C.

On the circle we have

$$|p|^2 = pp^* = \omega_0^2, \quad \text{or } p^* = \omega_0^2/p. \tag{58}$$

But we have already seen that when the complex zeros and poles occur in conjugate pairs we must have $[F_i(p)]^* = F_i(p^*)$. Hence *on the circle*

$$F_i(\omega_0^2/p) = F_i(p^*) = [F_i(p)]^* = V_i - i\Psi_i. \tag{59}$$

This is the value of the transformed function as we approach the circle from points just outside C, corresponding to the value $V_i + i\Psi_i$ for points just inside C. Thus the potential is continuous across C and consequently we have proved that for all points outside C, the function

$$F_e(p) = F_i(\omega_0^2/p) = V_e(p) + i\Psi_e(p) \tag{60}$$

is the exterior function for the circle.

We have just seen that on the circle $\Psi_e = -\Psi_i$, hence equation (24) for the integrated charge reduces to

$$q(\vartheta) = \frac{1}{\pi}[\Psi_i(\vartheta) - \Psi_i(\vartheta_0)] + Q_0, \tag{61}$$

where Q_0 is a constant charge density, and the charge is measured from ϑ_0.

12. Conformal Transformations

From the network point of view, unfortunately, the simple solution for a circular contour does not always lead to the best solution of the design problem. Hence we must also consider more general contours. The potential analogue method requires an *ab initio* choice of contour on which the zeros and poles of the approximating transmission function are to be located. Small changes in the contour shape should not be of great importance, but it may happen that our initial choice leads to a very complicated network when a much simpler one would satisfy the physical requirements. Experience is required to make the most effective use of the method, and various simplifications may frequently be available. For instance, it may be possible to split the assigned gain or phase functions into components, for some of which the zeros and poles may be located by inspection. Then the potential analogue is used to synthesize the remaining components.

There is, however, one limitation on the choice of contour which is inherent in the potential interpretation, namely, that the transmission function must be finite and analytic inside the contour. This is because the value of the potential *on* C defines its values at all points inside C only if these values, and their derivatives, are finite throughout the interior. We can see this intuitively when we remember that for a given charge distribution the potential and its derivatives are finite at all points not occupied by the charges.

It happens that the type of contour most frequently used up to the present has been the ellipse, and we shall discuss this contour in more detail later. For the present we shall consider more generally any simple closed contour in the complex p-plane, surrounding the frequency band of interest, $|\omega| < \omega_0$. The contour must be symmetric in the real p-axis, as we have seen, but we shall not impose any other restrictions except the fundamental one that the given complex potential must be analytic inside C.

We now introduce the theory of conformal transformations, and use the fundamental property that any simple closed curve in the finite p-plane may be mapped, by an analytic transformation, on the unit circle in a second complex plane, which we shall denote by the w-plane. Suppose that

$$p = \Gamma(w) \tag{62}$$

is such a transformation. Primarily the transformation must be such that points on the contour C in the p-plane become points on the unit circle, C_1, in the w-plane, but this is not sufficient to define Γ uniquely. To make the definition unique, in a way which we shall find convenient in solving our potential problem, we impose the following conditions:

$$\left.\begin{array}{l}
\text{1) } \Gamma(w) \text{ maps } C_1 \text{ on } C \\
\text{2) } \Gamma(w) \text{ maps the exterior of } C_1 \text{ on the exterior of } C \text{ in a one-to-one} \\
\quad\ \text{analytic manner} \\
\text{3) The point at infinity in the } w\text{-plane corresponds to the point at} \\
\quad\ \text{infinity in the } p\text{-plane} \\
\text{4) } \Gamma(+1) \text{ is real and positive.}
\end{array}\right\} \tag{63}$$

Now if our assigned transmission function in the p-plane is

$$F_i(p) = V_i(p) + i\Psi_i(p) \tag{64}$$

the assigned transmission function in the w-plane is

$$F_i'(w) = F_i[\Gamma(w)] = V_i'(w) + i\Psi_i'(w) \tag{65}$$

and our problem is to find the exterior function $F_e'(w)$ in the w-plane. Unfortunately this problem cannot usually be solved by the simple inversion theorem for the circle in the p-plane, because the transformation (62) introduces singularities in $F_i'(w)$ which are in addition to the singularities due to the poles and zeros of the original function. The second condition of the set (63) requires that $F_e'(w)$ must be analytic outside C_1, but in general $F_i'(w)$ is not analytic inside C_1 and the inversion theorem will therefore not lead to an analytic form for $F_e'(w)$. The second condition of the set (63) was deliberately chosen to make the mapping $F_e'(w)$ of the *unknown* exterior function $F_e(p)$ analytic outside C_1. The extra complexity of the potential problem for the general contour C, as compared with the circle in the p-plane, arises because it is not usually possible to define the transformation in such a way that, simultaneously, the mapping $F_i'(w)$ of the *known* interior function $F_i(p)$ is analytic inside C_1. Two exceptions are when $F_i(p)$ is constant so that $F_i'(w)$ is also constant (the equipotential distribution), and when $\Gamma(w)$ is a linear function (when the original contour in the p-plane is also circular).

Hence we must find a more general solution of the problem for the circle before we can use the potential analogue method to its fullest extent. This we shall do in the next section. For the moment let us assume that we have solved the problem for the unit circle in the w-plane, and thus determined a charge distribution on C_1 for which the potential is continuous across C_1. Now, by our definition of Γ, points on C_1 correspond to points on C. Hence we find the distribution on C by an inverse transformation in which the charge at any point on C_1 becomes the same charge at the corresponding point on C. This charge distribution on C has the required potential inside C. It may be simpler in practice to determine a convenient lumped charge distribution on C_1 and then transfer these lumped charges to the corresponding points on C.

It remains to determine $\Gamma(w)$, satisfying the conditions (63). One method is based on the remark above that if C is an equipotential in the p-plane then C_1 is an equipotential in the w-plane. Hence Γ might be defined as the transformation that maps equipotential distributions on C as equipotential distributions on C_1. This transformation has been determined for many contour shapes in the classical theory of equipotential distributions.

At the same time the precise shape of the contour is not usually critical for network purposes, so that it may be simpler to choose a $\Gamma(w)$ directly and determine the corresponding shape of the contour. A simple functional form involving two or three parameters might be assumed, for example,

$$\Gamma(w) = aw - \frac{b}{w} + \frac{c}{w^3} \tag{66}$$

where the parameters a, b, c will be sufficient to give C any length and breadth and a considerable further variation in shape. Illustrative shapes for transformations of the type (66) are shown in Fig. 15. In practice the special case of the ellipse, for which $c = 0$, is often adequate.

13. Poisson's Integrals

We turn now to a general solution of the exterior potential problem for the unit circle in the w-plane, which may be used when the simple inversion theorem is not applicable. For this purpose we start from Cauchy's integral,

$$F_e(w) = \frac{1}{2\pi i} \int_C \frac{F_e(\lambda)}{\lambda - w}\, d\lambda, \tag{67}$$

where C is a simple closed curve in the w-plane and the integration is taken clockwise round C. It is assumed that $F_e(w)$ vanishes at infinity at least as $1/w$, and then the integral expresses the value of an analytic function F_e at any point outside C in terms of its values on C.

We are interested particularly in applying (67) to a point just outside the unit circle in the w-plane. To do this we place the point w *on* the circle and then keep it just outside the actual contour by introducing an infinitesimal semicircular indentation as shown in Fig. 16. Over this semicircle the integral may be evaluated by writing $\lambda - w = \delta e^{i\alpha}$, where δ is the infinitesimal radius, and assuming that $F_e(\lambda)$ is practically constant; then its value is

$$\frac{1}{2\pi i} \int_{\alpha_1}^{\alpha_1+\pi} \frac{F_e(w)}{\delta e^{i\alpha}}\, i\delta e^{i\alpha}\, d\alpha = \tfrac{1}{2} F_e(w). \tag{68}$$

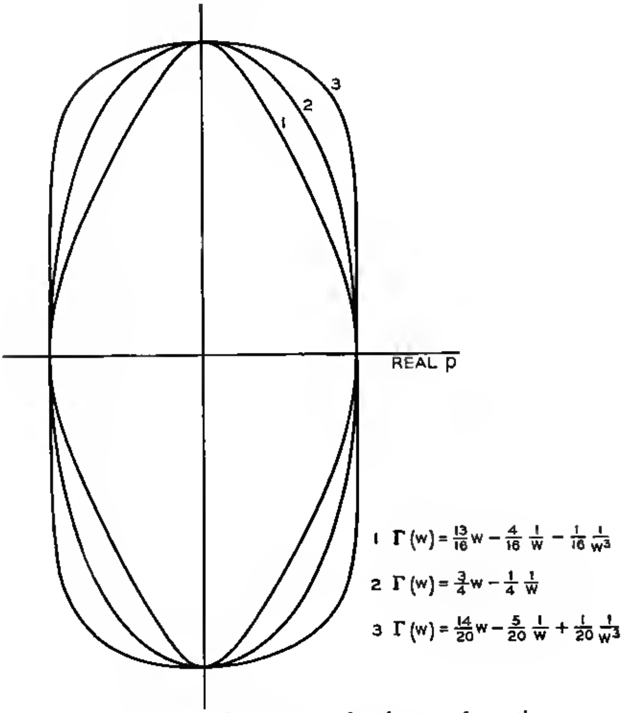

Fig. 15—Illustrative contours for the transformation.

$$p = \Gamma(w) = aw - \frac{b}{w} + \frac{c}{w^3}$$

Then over the contour C' which is the unit circle *excluding* the indentation, (67) becomes

$$F_e(w) = \frac{1}{\pi i} \int_{C'} \frac{F_e(\lambda)}{\lambda - w}\, d\lambda. \tag{69}$$

If we now interpret $F_e(\lambda)$ as the exterior complex potential of our charge distribution problem, on the circle, and introduce angular coordinates

$$\lambda = e^{i\vartheta}, \qquad w = e^{i\varphi}, \tag{70}$$

the integral may be written

$$F_e(\varphi) = -\frac{1}{\pi} P \int_0^{2\pi} \frac{V_e(\vartheta) + i\Psi_e(\vartheta)}{e^{i\vartheta} - e^{i\varphi}} e^{i\vartheta} \, d\vartheta, \tag{71}$$

where P denotes the principal value† of the integral, corresponding to the contour C'; that is, with an infinitesimal segment at the singularity $\vartheta = \varphi$

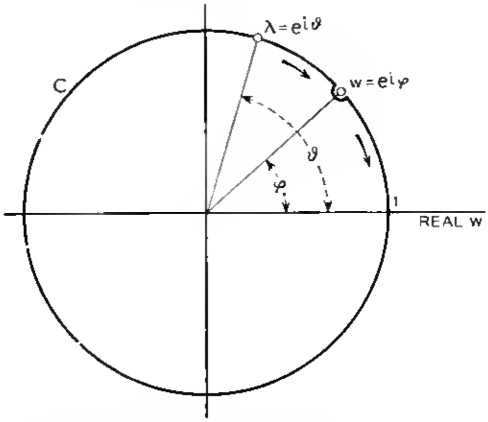

Fig. 16—Unit circle contour with semicircular indentation at p.

omitted. In the integral (71) the value of $V_e(\vartheta)$ is known (since it is equal to $V_i(\vartheta)$ on C') and we shall now show how to determine Ψ_e from V_e.

If we separate the real and imaginary parts of (71) we find

$$\begin{aligned} V_e(\varphi) &= -\frac{1}{2\pi} P \int_0^{2\pi} [V_e(\vartheta) + \Psi_e(\vartheta) \cot \tfrac{1}{2}(\vartheta - \varphi)] \, d\vartheta, \\ \Psi_e(\varphi) &= \frac{1}{2\pi} P \int_0^{2\pi} [V_e(\vartheta) \cot \tfrac{1}{2}(\vartheta - \varphi) - \Psi_e(\vartheta)] \, d\vartheta. \end{aligned} \tag{72}$$

Since we have assumed that $F_e(w)$ vanishes at infinity the integrals of V_e and Ψ_e round the circle will be zero, and (72) reduces to

$$\begin{aligned} V_e(\varphi) &= -\frac{1}{2\pi} P \int_0^{2\pi} \Psi_e(\vartheta) \cot \tfrac{1}{2}(\vartheta - \varphi) \, d\vartheta \\ \Psi_e(\varphi) &= \frac{1}{2\pi} P \int_0^{2\pi} V_e(\vartheta) \cot \tfrac{1}{2}(\vartheta - \varphi) \, d\vartheta. \end{aligned} \tag{73}$$

Further it is easy to verify that

$$P \int_0^{2\pi} \cot \tfrac{1}{2}(\vartheta - \varphi) \, d\vartheta = 0, \tag{74}$$

† E. T. Whittaker and G. N. Watson, Modern Analysis, Cambridge, 1920, p. 75.

so that the integrals in (73) will not be altered if we introduce a constant multiple of $\cot \frac{1}{2}(\vartheta - \varphi)$ in each integrand. This enables us to replace the improper integrals by convergent integrals, and find *Poisson's integrals* in a form particularly well adapted to the network problem:

$$\begin{aligned} V_e(\varphi) &= -\frac{1}{2\pi} \int_0^{2\pi} [\Psi_e(\vartheta) - \Psi_e(\varphi)] \cot \tfrac{1}{2}(\vartheta - \varphi)\, d\vartheta, \\ \Psi_e(\varphi) &= \frac{1}{2\pi} \int_0^{2\pi} [V_e(\vartheta) - V_e(\varphi)] \cot \tfrac{1}{2}(\vartheta - \varphi)\, d\vartheta. \end{aligned} \tag{75}$$

It may be helpful to engineers to note the similarity between these integrals and well-known integrals connecting gain and phase, which are of course the real and imaginary parts of complex transmission functions. Actually, the only essential difference is the shape of the contour on which the relations hold—here a circle, as opposed to the imaginary p-plane axis for the gain-phase relations.

Poisson's equations analogous to (75) may be found for points outside the unit circle by separating the real and imaginary parts of the original integral (67). The resulting integrals are convergent and there is no need to modify the integrands nor to indent the contour.

14. Use of the Inversion Theorem for Non-Circular Contours

We have seen that in the w-plane the interior function $F_i'(w)$ is not in general analytic inside C_1, so that the inversion theorem cannot be used directly. In other words, if $F_i'(w)$ has singularities inside C_1 then $F_i'(1/w)$ will have singularities outside C_1 and therefore cannot be the exterior potential $F_e'(w)$.

Thus, in general, we may have to use Poisson's integral to determine the exterior stream function. The inversion theorem may still be applied, however, if it is possible to separate the interior function into two parts, one of which, F_a, is analytic inside C_1, while the other, F_b, is analytic outside C_1. We write

$$F_i'(w) = F_a(w) + F_b(w), \tag{76}$$

and note that $F_a(1/w)$ is analytic outside C_1. Since $F_b(w)$ is also analytic outside C_1 the exterior function is given immediately by

$$F_e'(w) = F_a(1/w) + F_b(w), \tag{77}$$

where the transformation has to be applied only to F_a.

This represents at times a real simplification of the charge distribution problem, since $F_b(w)$ is the same on both sides of the contour and therefore

does not contribute to the discontinuity in the stream function. In fact the charge distribution on C_1 is now determined by

$$q'(\vartheta) = \frac{1}{\pi}[\Psi_a(\vartheta) - \Psi_a(\vartheta_0)] + Q_0', \tag{78}$$

where Q_0' represents a constant charge density in the w-plane, and Ψ_a is that part of the stream function on the circle contributed by $F_a(w)$.

Certain functions $F_i(p)$ lead to very simple separation formulas for any contour shape, provided $\Gamma(w)$ has been expressed in analytic form. A simple example is the linear phase function,

$$F_i(p) = -Kp, \tag{79}$$

for which

$$F_i'(w) = -K\Gamma(w). \tag{80}$$

By the definition of $\Gamma(w)$ this function has a pole at infinity, but is otherwise analytic outside C_1. Inside C_1 it will have poles at any poles of $\Gamma(w)$. We can separate out that part of $F_i'(w)$ which is analytic inside C_1 by considering the value of the derivative $d\Gamma/dw$ at $w = \infty$. This will have a finite value Γ_∞', and we write

$$F_i'(w) = (-K\Gamma_\infty')w + [-K\Gamma(w) + (K\Gamma_\infty')w]. \tag{81}$$

The first factor is analytic inside C_1 and the second outside C_1, hence the exterior function is

$$F_e'(w) = -\frac{K\Gamma_\infty'}{w} + [-K\Gamma(w) + (K\Gamma_\infty')w], \tag{82}$$

while at the charge point $w = e^{i\vartheta}$ the integrated charge is

$$q'(\vartheta) = -\frac{K\Gamma_\infty'}{\pi}\sin\vartheta + Q_0'. \tag{83}$$

A more general example of the use of the separation theorem will be found in the next section.

15. Elliptic Contours

The unit circle C_1 in the w-plane is mapped on the p-plane ellipse C of Fig. 17 by the transformation

$$p = \Gamma(w) = \tfrac{1}{2}\omega_0\left(\frac{w}{k} - \frac{k}{w}\right), \tag{84}$$

where the major axis of the ellipse is along the real frequency axis with foci at $\pm i\omega_0$, the intercepts on the ω-axis are at $\pm\frac{1}{2}i\omega_0(\frac{1}{k} + k)$, and the intercepts on the ξ-axis are at $\pm\frac{1}{2}\omega_0(\frac{1}{k} - k)$. This transformation will map the outside of C_1 on the outside of C if ω_0 and k are real positive constants with

$k < 1$. The eccentricity of the ellipse varies with k; in the limit $k \to 1$ the ellipse degenerates to the segment of the real frequency axis $|\omega| < \omega_0$.

Now for a given transmission function inside C, $F_i(p)$, the complex potential inside C_1 is $F_i'(w) = F_i[\Gamma(w)]$. In general this function will have singularities inside C_1, but when $F_i(p)$ may be expanded in a power series in p we may use the separation theorem of the last section to obtain a simple formula for the charge distribution on C_1. For instance, let $F_i(p)$ be a polynomial in p,

$$F_i(p) = \sum_n a_n p^n, \tag{85}$$

then

$$F_i'(w) = \sum_n a_n \left(\frac{\omega_0}{2}\right)^n \left[\frac{w}{k} - \frac{k}{w}\right]^n. \tag{86}$$

When the binomial is expanded in a power series the terms involving positive

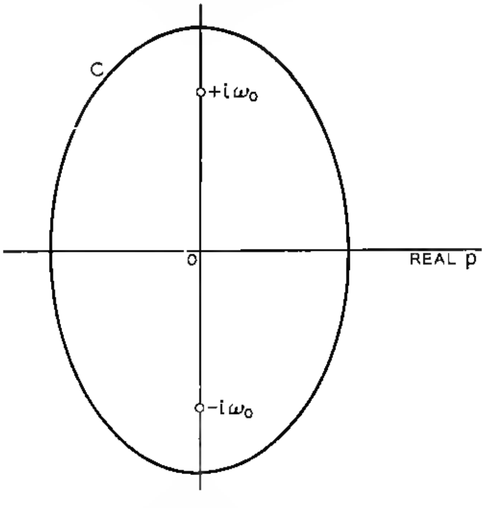

Fig. 17—Elliptic contour in the p-plane.

powers of w will belong to $F_a(w)$, while the terms involving negative powers will belong to $F_b(w)$. Hence the parts of $F_i'(w)$ analytic respectively inside and outside C_1 are

$$F_a(w) = \sum_n a_n \left(\frac{\omega_0}{2}\right)^n \left[\left(\frac{w}{k}\right)^n - n\left(\frac{w}{k}\right)^{n-2} + \frac{n(n-1)}{2!}\left(\frac{w}{k}\right)^{n-4} - \cdots\right]$$

$$F_b(w) = \sum_n a_n \left(\frac{\omega_0}{2}\right)^n \left[\left(-\frac{k}{w}\right)^n - n\left(-\frac{k}{w}\right)^{n-2} + \frac{n(n-1)}{2!}\left(-\frac{k}{w}\right)^{n-4} - \cdots\right]. \tag{87}$$

When n is odd each series ends in the first power of its argument; when n is even F_a ends in a constant (which may be ignored in determining the charge distribution) while F_b ends in a term in w^{-2}.

We have seen that the charge distribution on C_1 is determined by $F_a(w)$, and from equation (78) we find

$$q'(\vartheta) = \frac{1}{\pi}\sum_n a_n\left(\frac{\omega_0}{2}\right)^n\left[\frac{\sin n\vartheta}{k^n} - n\frac{\sin (n-2)\vartheta}{k^{n-2}} + \cdots\right]. \tag{88}$$

Corresponding to each power p^n in $F_i(p)$ we have a finite Fourier sine series for $q'(\vartheta)$. Conversely, the powers of p from 0 to n, for each value of n, may be summed in such proportions that the resulting n^{th} degree polynomials, $F_i(p)$, correspond to charge distributions $\sin n\vartheta$ on C_1. The actual form of these polynomials may be determined by considering the formulas we have just derived.

If the charge distribution is $C_n\dfrac{\sin n\vartheta}{\pi k^n}$, the corresponding term in $F_a(w)$ is $C_n(w/k)^n$, and this is matched by the term $C_n(-k/w)^n$ in $F_b(w)$. Hence the interior function for this charge is

$$F_i'(w) = C_n\left[\left(\frac{w}{k}\right)^n + \left(-\frac{k}{w}\right)^n\right]. \tag{89}$$

Now *on the real frequency axis*, $p = i\omega$, the solution of equation (84) for w in terms of ω is

$$w = ke^{i\delta}, \qquad \delta = \sin^{-1}\frac{\omega}{\omega_0}. \tag{90}$$

This means that the real frequency axis in the p-plane in the region $|\omega| < \omega_0$ corresponds to a semicircle of radius k in the w-plane. Substituting from (90) in (89) we have

$$F_i(i\omega) = C_n[e^{in\delta} + (-)^n e^{-in\delta}]. \tag{91}$$

Hence, corresponding to a charge distribution

$$q'(\vartheta) = \sum_{n=1}^{\infty} C_n\frac{\sin n\vartheta}{2\pi k^n} + Q_0'$$

in the w-plane, we have, on the real frequency axis in the p-plane.

$$\begin{aligned} F_i(i\omega) &= \frac{1}{2}\sum_{n=1}^{\infty} C_n[e^{in\delta} + (-)^n e^{-in\delta}] + C_0 \\ &= \sum_{m=0}^{\infty} C_{2m}\cos 2m\delta + i\sum_{m=0}^{\infty} C_{2m+1}\sin (2m+1)\delta \end{aligned} \tag{92}$$

We write this result alternatively in the form

$$F_i(i\omega) = \sum C_{2m} T_{2m}(\omega) + i\sum C_{2m+1} T_{2m+1}(\omega) \tag{93}$$

where T_{2m} is the Tchebycheff polynomial of even order,

$$T_{2m}(\omega) = \cos [2m \sin^{-1}(\omega/\omega_0)] \tag{94}$$

and T_{2m+1} may be interpreted as a modified Tchebycheff polynomial of odd order, particularly adapted to network synthesis problems,

$$T_{2m+1}(\omega) = \sin [(2m + 1) \sin^{-1}(\omega/\omega_0)]. \tag{95}$$

It is easy to verify that the T's are in fact polynomials in ω/ω_0. For the first few values of n we find

$$\begin{gathered} T_0 = 1, \qquad T_1 = \frac{\omega}{\omega_0}, \qquad T_2 = 1 - 2\left(\frac{\omega}{\omega_0}\right)^2 \\ T_3 = 3\frac{\omega}{\omega_0} - 4\left(\frac{\omega}{\omega_0}\right)^3, \qquad T_4 = 1 - 8\left(\frac{\omega}{\omega_0}\right)^2 + 8\left(\frac{\omega}{\omega_0}\right)^4, \text{ etc.} \end{gathered} \tag{96}$$

In dealing with prescribed gain and phase functions for elliptic contours, the simplest procedure is to expand the gain, not in an even power series, but in a series of even Tchebycheff polynomials, while the phase is expanded in a series of odd Tchebycheff polynomials. Such expansions are always possible for analytic functions, and it should be pointed out that their region of convergence is greater than that for a simple power series. An additional advantage of using the polynomials instead of the power series is that the T's are orthogonal in the frequency range $|\omega| < \omega_0$, while the various terms of the power series are not. This increases the rapidity of convergence and leads to a more efficient solution of the design problem.

A simple illustration of the effect of contour shape on the accuracy of the lumped charge approximation to the transmission function is shown in Fig. 18. This refers to the constant gain filter we discussed, for a circular contour, in Section 10. The granularity error for the circle (curve 1) is very small at low frequencies, while for the two ellipses (curves 2 and 3) it is small, but oscillatory, and the oscillations become larger as the ellipse becomes narrower. On the other hand, at frequencies near the upper limit ω_0 of the frequency band, the granularity error is much smaller for the ellipses than for the circle; in other words, the cut-off frequency is more sharply defined.

16. The Expansion Theorem for General Contours

The term by term correspondence between the Fourier expansion of the charge on C_1 and the expansion of the gain and phase functions as series of polynomials holds also for general contour shapes. In the general case

the polynomials are not of the Tchebycheff type, and as a rule they are not orthogonal.

By its definition in (63) $\Gamma(w)$ can always be expanded in a series of the form

$$\Gamma(w) = \Gamma'_\infty w + g_0 + \sum_n g_n w^{-n}, \tag{97}$$

valid on and outside C_1. It follows that p^n, which transforms into $[\Gamma(w)]^n$, can always be expanded as an n^{th} degree polynomial in w plus a power series in $1/w$, and these correspond to $F_a(w)$ and $F_b(w)$ respectively. The charge

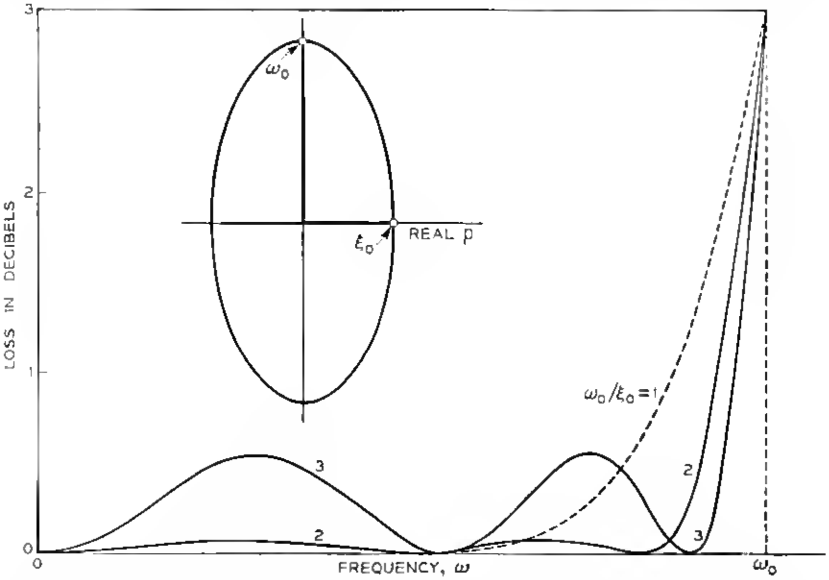

Fig. 18—Illustrating the effect of contour shape on the accuracy of the approximate transmission function for a flat filter.

on C_1 corresponding to p^n is determined by $F_a(w)$ and is therefore a finite Fourier sine series, similar to (88) except for more general coefficients. Conversely, we can always construct a polynomial in p of degree n, by choosing appropriate coefficients for the various powers of p, in such a way that the charge on C_1 is merely sin $n\vartheta$. In other words if

$$q'(\vartheta) = \sin n\vartheta$$

then

$$F_r(p) = P_{\Gamma n}(p)$$

where $P_{\Gamma n}(p)$ is a polynomial of degree n whose coefficients depend only on $\Gamma(w)$, that is on the shape of the contour.

By summing the above relations for all values of n we have the general expansion theorem,

$$\begin{aligned} F_i(p) &= \sum C_n P_{\Gamma n}(p), \\ q'(\vartheta) &= \sum C_n \sin n\vartheta. \end{aligned} \tag{98}$$

Thus if the assigned gain and phase functions can be expanded in terms of the polynomials $P_{\Gamma n}(p)$, appropriate to the given contour, then the Fourier expansion of the charge on C_1 can be written down immediately.

17. High-pass and Band-pass Filters

So far we have assumed that the contour in the p-plane is a simple closed curve. This is adequate as long as the positive frequencies of interest extend

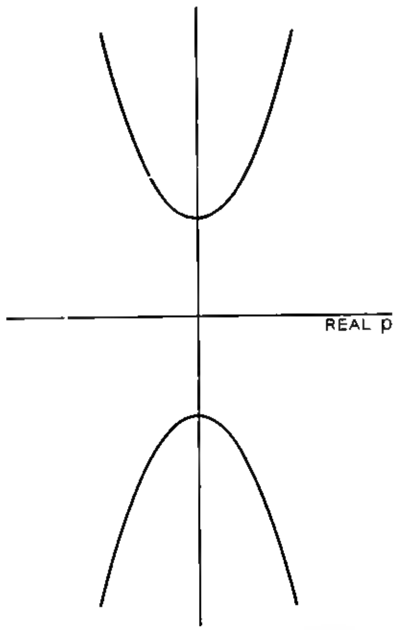

Fig. 19—Appropriate contour for a high-pass filter.

from zero to a finite upper bound, ω_0, as in low-pass filters. For high-pass filters, in which the positive frequencies extend from a lower bound, ω_0, to infinity, an appropriate shape of contour is shown in Fig. 19. However, high-pass problems can always be reduced to the low-pass type by simply using $1/p$ as the variable instead of p.

In band-pass filters, whose positive frequencies of interest extend between two finite values, $\omega_0 < \omega < \omega_1$, we must be able to use a contour of the type shown in Fig. 20a. This consists of two disjoint closed curves, one above and one below the real axis (real p). The physical requirements are satisfied if the curves are symmetric about the real p-axis, but as usual it is advantageous to make them symmetric also about the real ω-axis. For then, if a point p_ν lies on one of the curves, the point $-p_\nu$ will lie on the other. This makes it possible to map the disjoint contour C on a single closed curve C_2 in the p^2-plane, the y-plane of Fig. 20b, by means of the transformation $p = \sqrt{y}$. The single contour C_2 may now be mapped on the unit circle in

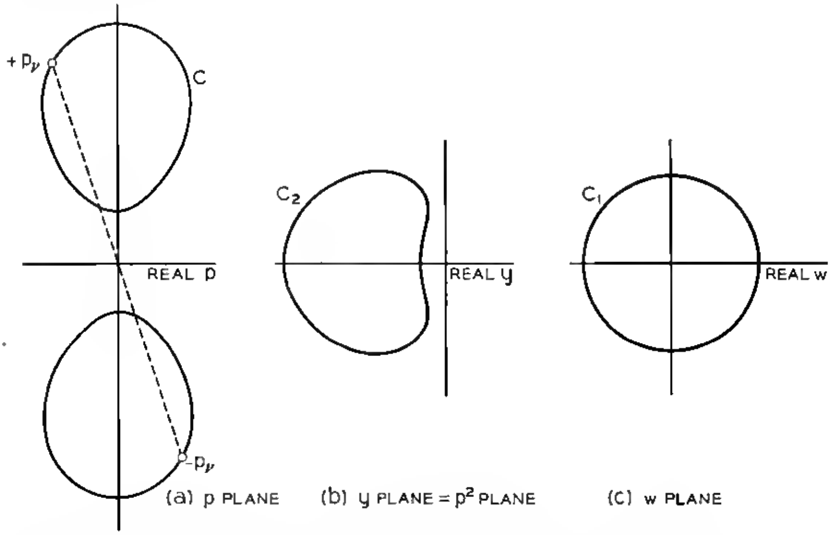

Fig. 20—Contours for a band-pass filter; (a) disjoint contour symmetric about both axes in p-plane, (b) single contour in p^2-plane, (c) unit circle in w-plane.

the w-plane, Fig. 20c, by means of a second transformation $y = \Gamma_1(w)$. Combining the transformations we have

$$p^2 = \Gamma_1(w), \qquad p = \sqrt{\Gamma_1(w)} \tag{99}$$

as the transformation which maps C on C_1.

The conditions on the function $\Gamma_1(w)$ are the same as in (63) except that, since C_2 is in the left half of the y-plane and does not cut the positive real axis, the fourth condition must be replaced by a similar requirement.

$$\Gamma_1(+1) - \Gamma_1(-1) \text{ is real and positive.}$$

Now the presence of the square root in the transformation (99) may introduce branch points in the w-plane corresponding to the branch points

at zero and infinity in the y-plane. There will be no branch points if the original transmission function $F_i(p)$ is an even function of p, for then the exterior function $F_e(p)$ will also be an even function. In this case the simple closed curve analysis does not have to be modified. The usual method can be used to determine $\Psi_e'(w)$ in the w-plane, and the charge distribution on C_1 determined.

When $F_i(p)$ is an odd function, however, we have to proceed more carefully, since the transformation now introduces branch points in the w-plane corresponding to a *factor* $\sqrt{\Gamma_1(w)}$. In this case we assume that $\Gamma_1(w)$ is given in analytic form, and determine the root w_1 of $\Gamma_1(w) = 0$ which lies outside C_1. Then it will be possible to express $F_e'(w)$ in the form

$$F_e'(w) = \frac{G(w)}{\sqrt{1 - w/w_1}} \tag{100}$$

where $G(w)$ is analytic outside C_1, and has the proper behavior at infinity. From the conditions imposed on Γ_1 it can be shown that w_1 is real; hence we introduce a rationalizing factor

$$M(w) = \sqrt{\left(1 - \frac{w}{w_1}\right)\left(1 - \frac{1}{ww_1}\right)}, \tag{101}$$

and multiply both sides of equation (100) by $M(w)$. This leads to

$$M(w)F_e'(w) = \sqrt{1 - \frac{1}{ww_1}}\, G(w) = H(w), \tag{102}$$

where $H(w)$ is analytic outside C_1. On C_1, $|w| = 1$, so that $M(w)$ is real and *on* C_1 the potential and stream functions are defined by

$$\begin{aligned} M(w)\, V_e'(w) &= \operatorname{Re} H(w), \\ M(w)\, \Psi_e'(w) &= \operatorname{Im} H(w). \end{aligned} \tag{103}$$

Thus the real part of $H(w)$ is determined by the known potential $V_e'(w)$; this determines in turn the imaginary part of $H(w)$ and hence $\Psi_e'(w)$ is determined.

When $F_i(p)$ is neither even nor odd we divide it into even and odd parts and treat each part separately. If only the gain is important we need retain only the even part, or if only the phase is important we consider only the odd part.

18. Examples

So far we have been describing the potential analogue method in general terms, and developing a systematic design procedure applicable to a wide range of problems. The method involves a certain arbitrariness, in the initial choice of contour, and there may also be some doubt in the reader's mind as

to the accuracy of the final result, since a general theory of granularity errors has not been developed. Hence in this section we shall consider the application of the method to some actual engineering problems. This should aid the reader in using the method himself, and should also help to convince him of its validity.

Example 1. The Gaussian Filter

It is required to design a low-pass filter whose voltage transfer ratio is $\exp(-b\omega^2)$ and which has constant phase delay in the prescribed frequency range. For convenience we choose our unit of frequency to make the cut-off frequency equal to unity, and then we choose our contour C to be an ellipse in the p-plane passing through the points $p = \pm\frac{1}{2}, \pm i$.

The assigned transmission function in the p-plane is

$$F_i(p) = bp^2 - \beta p,$$

and the transformation which maps C on the unit circle in the w-plane is

$$p = \Gamma(w) = \frac{3w}{4} - \frac{1}{4w}.$$

In the w-plane the transmission function is

$$F_i'(w) = b\left(\frac{9w^2}{16} - \frac{3}{8} + \frac{1}{16w^2}\right) - \beta\left(\frac{3w}{4} - \frac{1}{4w}\right),$$

and the part of F_i' analytic inside C_1 is

$$F_a(w) = \frac{9b}{16}w^2 - \frac{3\beta}{4}w - \frac{3}{8}b.$$

Hence, by the separation theorem, the required continuous charge distribution on C_1 is

$$q'(\vartheta) = \frac{9b}{16\pi}\sin 2\vartheta - \frac{3\beta}{4\pi}\sin\vartheta + \frac{Q\vartheta}{2\pi},$$

where we have assumed a total charge Q on the circle.

In practice the values of b and Q are usually assigned, while the magnitude of the phase delay is at our disposal. Hence we choose β large enough to insure that $q'(\vartheta)$ is a monotonic decreasing function for $0 < \vartheta < \frac{\pi}{2}$. This makes it possible to divide the continuous charge into a set of unit steps, such that these steps are negative in the right half plane, and therefore correspond to zeros of the transmission function. A typical set of numerical values is

$$b = \frac{4}{3}, \qquad Q = 3, \qquad \beta = \frac{13}{3}\pi.$$

For these values we find unit increments in q' at the zeros (negative steps) 0, $\pm 25^\circ.47$, $\pm 55^\circ.76$; and at the poles (positive steps) $\pm 120^\circ.65$, $\pm 142^\circ.89$, $\pm 158^\circ.99$ and $\pm 173^\circ.17$. These five zeros and eight poles on the unit circle in the w-plane are now mapped back to the corresponding points on the ellipse in the p-plane, where they give the location of the zeros and poles of the approximate transmission function. Figure 21 illustrates the accuracy of the resulting approximation to the prescribed gain and phase.

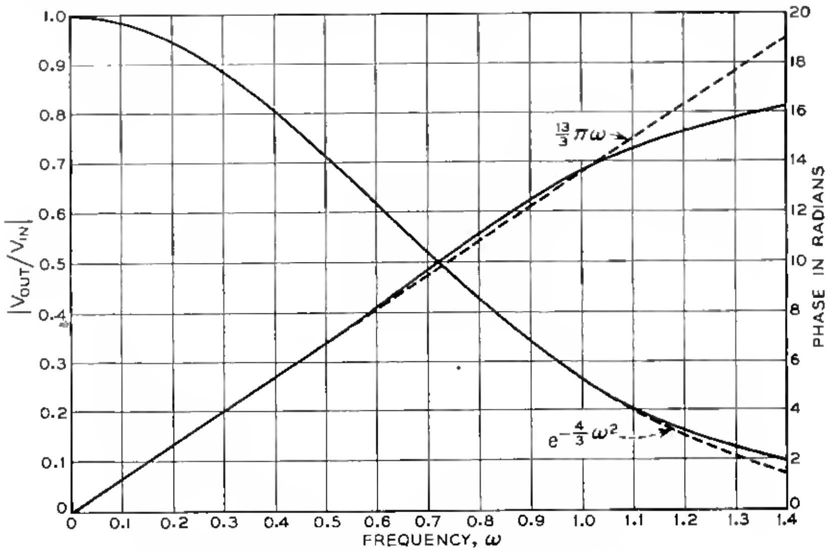

Fig. 21—Gain and phase curves for the Gaussian filter.

Example 2. The Coaxial Cable Equalizer

A section of coaxial line of finite conductivity has an insertion loss proportional to $\sqrt{\omega}$. The problem is to design a network which will equalize this distortion, that is, a network which has a transmission function

$$F_t(p) = k\sqrt{p}$$

in the frequency range $|\omega| < 1$.

This example is included partly because of its engineering importance, but also because it gives us the opportunity to introduce a particular type of contour, the *equipotential contour*. This consists of fitting the contour C to an *equipotential* of the transmission function, except for an arc at infinity (if C were everywhere equipotential $F_t(p)$ could only be constant). Thus the contour is not closed in the finite part of the plane, but is supposed to be closed through an arc at infinity so chosen that the charges on this arc will not produce any appreciable effect in the finite part of the plane.

For the cable function we introduce polar coordinates, $p = \rho e^{i\varphi}$, in the p-plane, so that

$$F_i(p) = k\rho^{1/2} e^{i\varphi/2},$$

$$V_i(p) = k\rho^{1/2} \cos \tfrac{1}{2}\varphi, \qquad \Psi_i(p) = k\rho^{1/2} \sin \tfrac{1}{2}\varphi,$$

and it is easy to see that the equipotentials are parabolas in the p-plane, as illustrated in Fig. 22a. Along the equipotential $V_i = k\sqrt{a}$ the stream function is

$$\Psi_i(p) = \pm k\sqrt{\rho - a}$$

where the positive sign refers to that part of the parabola which lies above the real p-axis and the negative sign to the part below the real p-axis. The closure of the contour at infinity is shown in Fig. 22b.

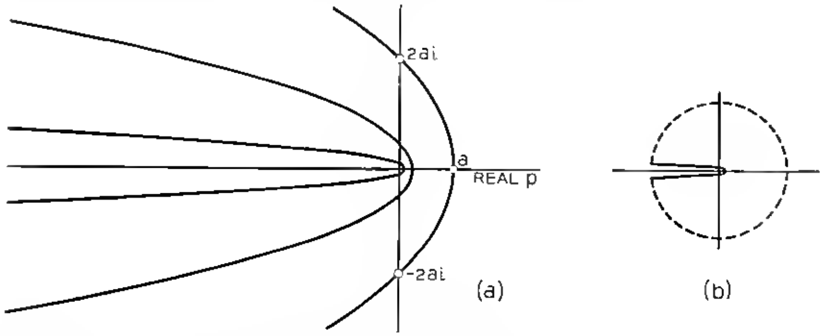

Fig. 22—The cable transmission function $K\sqrt{p}$; (a) equipotential contours are parabolas, (b) contour closed at infinity through a circular arc.

If charge is placed on the equipotential in such a way that $\Psi_i/2\pi$ represents the integrated charge density, then the correct potential and stream function will be produced everywhere to the right of the contour and the potential to the left of the contour will be the constant $k\sqrt{a}$. To keep the contour from crossing the Im p-axis we must take $a = 0$. Then the parabola degenerates into the negative real p-axis and charge is distributed with integrated density function

$$Q(\rho) = -\frac{k}{\pi}\sqrt{\rho}$$

on the axis.

The lumped charge approximation consists of placing zeros at points $p_n = -\rho_n$ where $Q(\rho_n) = n - \frac{1}{2}$; *i.e.* zeros are to be placed at

$$p_n = -(n - \tfrac{1}{2})^2 \frac{\pi^2}{k^2}, \qquad n = 1, 2, 3 \cdots$$

The gain and phase for this infinite array of zeros are obtained from the function

$$\log \prod_{n=1}^{\infty} \left[1 + \frac{pk^2}{(n - \frac{1}{2})^2\pi^2} \right] = \log \cos (ikp^{1/2})$$
$$= kp^{1/2} + \log (1 + e^{-2kp^{1/2}}) + \text{const.}$$

Thus the correct function $kp^{1/2}$ is obtained modified by a term of the order $e^{-2kp^{1/2}}$ representing "granularity error".

The solution as it stands is impractical for three reasons:

(i) an infinite number of singularities are used.

(ii) the singularities are all zeros so that one cannot satisfy the physical realizability requirements.

(iii) the granularity error becomes appreciable at low frequencies.

Objections (ii) and (iii) may be avoided by choosing two numbers k_1, k_2 such that $k = k_2 - k_1$ and making lumped charge approximations for $k_2p^{1/2}$ and $k_1p^{1/2}$ separately. That is, we put zeros at $-(n - \frac{1}{2})^2\pi^2/k_2^2$ and poles at $-(n - \frac{1}{2})^2\pi^2/k_1^2$. By choosing k_1 and k_2 large enough we obtain a very fine-grained approximation to the ideal (continuous) charge distribution and can make the frequency at which granularity effects become bothersome as low as desired. Moreover since poles as well as zeros are used, we are now in a better position to satisfy the physical realizability requirements. When designing the network in this way it is convenient to make k_2/k_1 a rational number with numerator and denominator as small as possible. If the numerator and denominator are q_2 and q_1 then every zero p_n, for which $2n - 1$ is a multiple of q_2, is cancelled by a pole which falls at the same place.

The most obvious way to remedy defect (i) is to use just the first N zeros and the first N poles, picking N large enough so that the infinite set of zeros and poles which are being ignored produce only a negligible effect in the frequency band of interest $|\omega| < 1$. To get an idea of how large N must be, we evaluate the integral

$$f(p) = \int_R^{\infty} \frac{k \log (1 + p/r)}{2\pi \sqrt{r}} \, dr,$$

which represents the gain and phase contributed by all the charge from $p = -R$ to $p = -\infty$ in the continuous distribution. The substitution $r = x^2$ transforms the integral into an easily handled form and we find

$$f(p) = -\frac{k}{\pi} \left[\sqrt{R} \log \left(1 + \frac{p}{R} \right) - 2\sqrt{p} \tan^{-1} \sqrt{p/R} \right],$$

so that $f(p)$ is about $k\, p/\pi\sqrt{R}$ when $|R/p|$ is large.

In practice we soon find that we must use an unnecessarily large number N of zeros and poles to get good accuracy from the simple trick just described. A better plan is to keep just those zeros and poles which lie within some more moderate distance from the origin, say $R = 2$. Then the remaining gain and phase $f(p)$ must be approximated by other means. This offers no special difficulty; the disagreeable $p^{1/2}$ type singularity at the origin has already been produced, leaving $f(p)$ a relatively slowly varying function over the band $|\omega| < 1$. One way of approximating $f(p)$ by the log of a rational function with the desired number of zeros and poles is first to find a polynomial approximation to $e^{f(p)}$ and then pick the rational function which has the same first few terms in its power series as the polynomial. In the design carried out at BTL the polynomial approximation was performed by a method using Tchebycheff polynomials. This method will be the subject of a later paper. For purposes of illustration we may equally well imagine $f(p)$ to be produced by placing charge on an elliptic contour surrounding the interval $|\omega| < 1$.

The following numerical example will give the reader some idea of how well the method works in actual practice. The cable had a loss of 5.368 nepers (46 db) at $\omega = 1$ and it was required that the cable be equalized to within .005 db from $\omega = .02$ to $\omega = 1$. Using zeros only on the negative real axis, the granularity error would have been much too high. Sufficiently low granularity error was obtained by putting poles at

$$p = -.0068498\ (2n - 1)^2$$

and zeros at

$$p = -.0034948\ (2n - 1)^2.$$

This choice of position of zeros and poles makes every seventh zero cancel every fifth pole. In the final design only 6 of these zeros and 6 of the poles were used. The remaining gain and phase were produced, to the desired accuracy, by a pair of real poles at $p = -1.5$ and four pairs of conjugate complex poles lying close to an elliptic contour about the frequency band of interest.

Example 3. Delay Equalizer

A problem of frequent occurrence is that of "delay equalizing" a given network with known singularities. From the potential analogue point of view the problem is, given the location and sign of certain lumped charges, to find a distribution $Q_1(s)$ of charge on a contour C which produces no other effect on the real frequency axis in the range of interest but to cancel the transverse component of the electric field of the given charges. The distribu-

tion of charge $Q_1(s)$ as it stands gives rise to non-physical networks with poles in the right half-plane. However it is possible to add to $Q_1(s)$ a distribution of charge producing a high enough uniform cross-axis field (flat delay) so that the total charge distribution $Q(s)$ yields physical networks.

For the time being consider just the equalization of one singularity. If we solve this simple problem the $Q_1(s)$ for the general case of any number of singularities can be obtained by adding up the charge distributions for the individual singularities. For the sake of concreteness imagine the singularity to be a unit positive charge at $p_0 = -a + ib$ in the left hand p-plane. What is needed is a distribution $q_1(s)$ of cbarge on C which produces inside the contour the complex potential

$$W = \frac{1}{2} \log \frac{p - p_0}{p + p_0^*}$$

corresponding to a charge $-\frac{1}{2}$ at p_0 and a charge $+\frac{1}{2}$ at $-p_0^*$. By the phase invariant transformation, these two charges give the same field across the ω-axis as a unit negative charge at p_0, while along the axis their fields cancel. Note that we have reversed the sign of the charge at p_0. This is because the shielding distribution on C due to any set of exterior charges must be such that its potential inside C exactly cancels the potential of the charges, that is, it matches the potential that would be obtained if the signs of all charges were reversed.

Now the complex potential of a point charge Q at $-p_0$, outside C, is $F(p) = -Q \log (p - p_0)$. When this is mapped on the w-plane by a transformation $p = \Gamma(w)$ which maps C on the unit circle C_1 the transformed function may be separated into two parts, analytic respectively inside and outside C_1,

$$F_a(w) = -Q \log (w - w_0), \qquad F_b(w) = -Q \log \frac{\Gamma(w) - \Gamma(w_0)}{w - w_0},$$

where w_0 is the w-plane mapping of p_0, defined by $p_0 = \Gamma(w_0)$, and w_0 is outside C_1. We have seen that the mapping of the charge distribution $q(p)$ on C into the charge distribution $q'(w)$ on C_1 is determined by $F_a(w)$, and in the present case $F_a(w)$ represents the complex potential of a point charge Q located at w_0. It follows that the required shielding distribution on C in the presence of exterior charges maps into the shielding distribution on C_1 in the presence of the mappings *exterior to* C_1 of the exterior p-plane charges.

Thus in the w-plane our equalization problem is to determine the shielding distribution on C_1 due to a charge $+\frac{1}{2}$ at w_0 and a charge $-\frac{1}{2}$ at $\overline{w}_0$, where $-p_0^* = \Gamma(\overline{w}_0)$. Since we are considering only one singularity in the p-plane, and ignoring the physical requirement of an equal singularity at the conjugate complex point, we cannot apply the simple form of the inversion

theorem to $F_a(w)$ directly. We could modify the theorem without difficulty but we may also solve the problem by using the well-known electrostatic method of images.† The complex potential for the required shielding distribution is

$$W'(w) = \frac{1}{2}\log\frac{w - w_0}{w - \dfrac{1}{w_0^*}} - \frac{1}{2}\log\frac{w - \bar{w}}{w - \dfrac{1}{\bar{w}^*}},$$

and the shielding charge distribution is obtained by evaluating the imaginary part of W'. If the contour C is symmetric with respect to the real frequency axis, symmetry considerations in the w-plane will show that $\bar{w} = -w_0^*$; then the charge distribution may be written in the explicit form

$$q_1(\theta) = \frac{1}{2\pi}\tan^{-1}\left[\frac{A(R^2 - 1)}{B(R^2 + 1) - 2R^2\sin\theta}\right],$$

where $w_0 = -A + iB$ and $R^2 = A^2 + B^2$. This (integrated) charge distribution, when mapped back on the original p-plane contour C, becomes the shielding distribution $q_1(s)$ sought. If the singularity were a zero instead of a pole, q_1 would be given by the same expression with a factor -1.

The procedure for delay equalizing a group of singularities can be outlined as follows:

(1) Find a conformal mapping of the outside of C on the outside of the unit circle.

(2) Compute

$$Q_1 = \sum_i q_{1i}$$

as a function of θ. Here the sum runs over all the given singularities and q_{1i} is the distribution which equalizes the i-th singularity (computed from an expression like that for q_1 given above).

(3) Since Q_1 puts some poles in the right half-plane, compute

$$Q = Q_1 - D\sin\theta,$$

choosing the constant D large enough to make all the poles of the distribution Q lie in the left half-plane. The only effect of the distribution $D\sin\theta$ is to add flat delay.

(4) Approximate Q by a function with unit steps, say at $\theta_1, \theta_2, \cdots, \theta_N$.

(5) Map the singularities found in (4) on the p-plane to obtain the equalizer singularities.

Figures 23, 24, and 25 illustrate a delay equalizer design taken from actual practice. Figure 23a shows the p-plane locations of the singularities

† L. Page, Introduction to Theoretical Physics, D. Van Nostrand and Co., New York, 1935, p. 404.

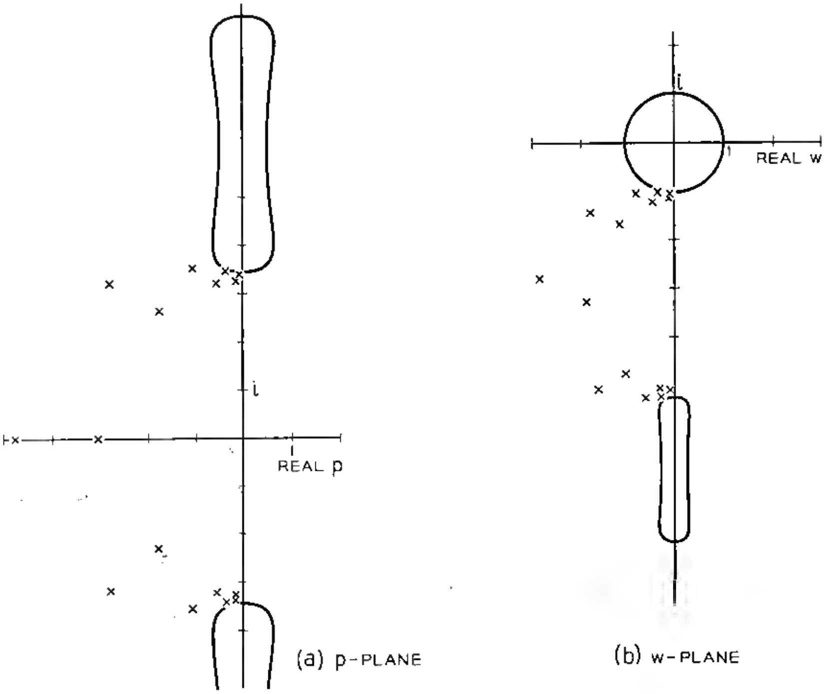

Fig. 23—Delay equalizer singularities and assigned contour; (a) in p-plane, (b) mapped on w-plane.

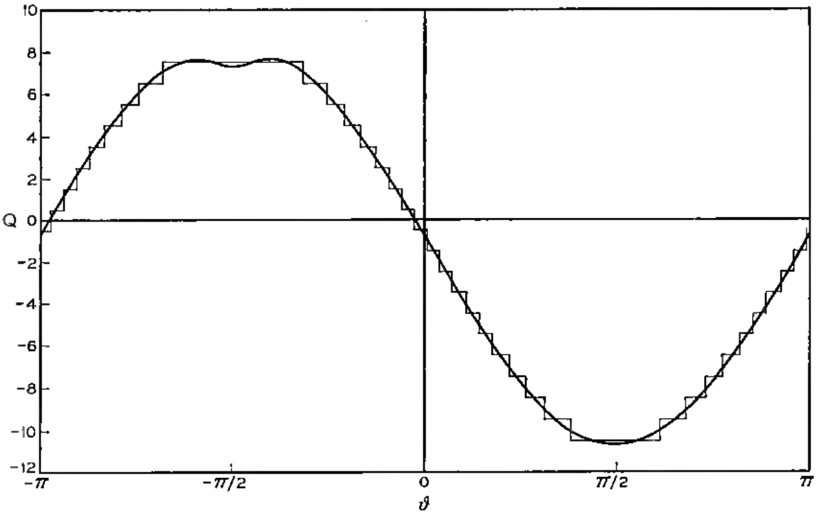

Fig. 24—Curve for integrated charge as function of θ and its approximation by step functions.

(all poles) of a high-pass filter.† The contour C is shown surrounding the band of interest. There are really two contours, one surrounding a band of positive frequencies and another surrounding a band of negative frequencies. To obtain an exact solution for the charge distribution using two contours would be very troublesome. Fortunately the two contours are far enough

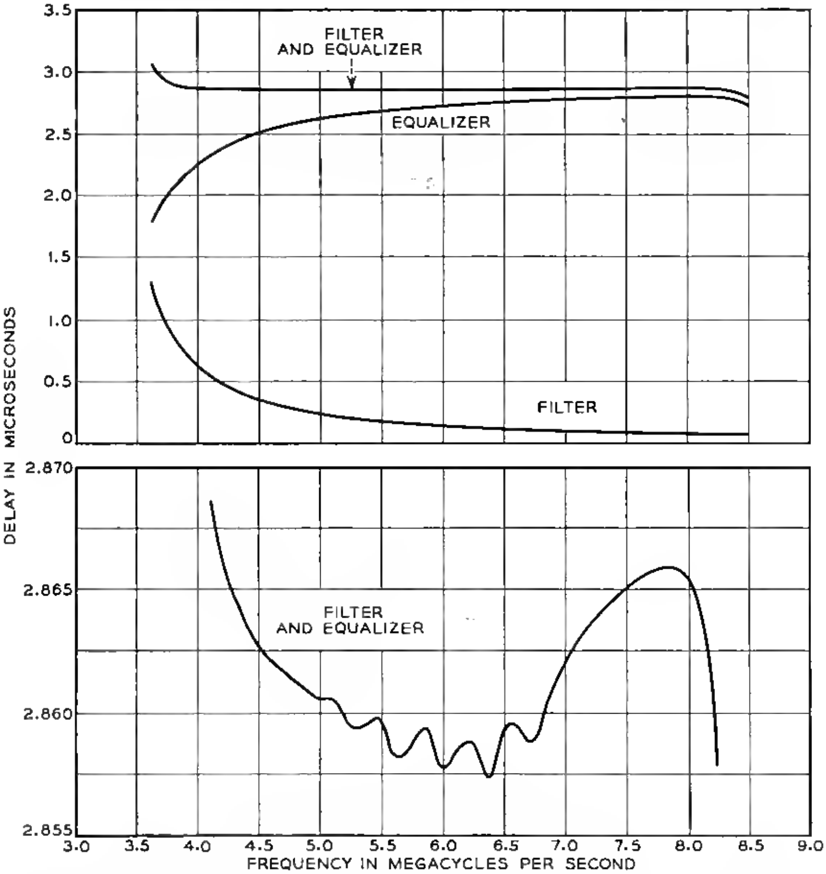

Fig. 25—Curves showing phase delay of filter, equalizer and their combination.

apart so that the charges on one produce only small effects inside the other. The charge distribution on the upper contour was found by replacing the lower contour charges by a single large pair of positive and negative charges.

The delay to be produced by the equalizer varies slowly across the band

† The zeros (not shown) of the filter are on the imaginary axis below the pass band. They are ignored because they contribute no delay.

except near the low-frequency end. In view of the success of the condenser plate contour for producing flat delay, it was felt that C should be chosen to be nearly rectangular. An actual rectangle could have been used for C, but tbe mapping to a circle involves unwieldy expressions containing elliptic functions. The contour shown was used instead because it is nearly rectangular and because it bas a simple mapping function

$$p = i\,6.10 + 1.775w - \frac{1.075}{w} - \frac{0.2}{w^3}$$

(here p is expressed directly in megacycles). This contour was obtained by plotting a few of the contours for different numerical values of the constants in the above mapping function. The w-plane images of the singularities and of the lower contour are shown in Fig. 23b.

The charge Q as a function of θ is shown in Fig. 24 together with the step function approximation. Figure 25 shows the delay produced by the filter and equalizer together.

19. Acknowledgements

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